Physical Determinacy and Mathematical Determinacy

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Outline

Section 1

Introduction

Overview

In this talk I will

- Review a Putnamian model theoretic worry about realism and reference
- Note that it's fashionable to:
  - take this worry seriously for mathematics
  - while accepting unapolegetic realism about physical objects and physical laws/possibility facts
- Argue this combination of views is untenable

Section 2

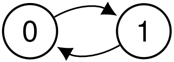
Nonstandard Models of Number Theory

Theories and Models I

- Not all mathematical theories determine a unique structure under consideration.
- Consider this simple observation about the natural numbers: Every number has a successor different than itself.

$$(\forall x)\,((S(x) \neq x) \qquad (S^1)$$

- In addition to the natural numbers this could be satisfied (modeled by) by the two element structure $0, 1$ with $S(0) = 1, S(1) = 0$ or represented graphically

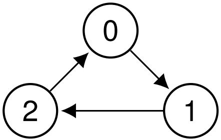


Theories and Models II

- Add the observation that the successor of the successor of x can't be x.

$$(\forall x)\,((S(S(x)) \neq x) \qquad (S^2)$$

- In addition to the natural numbers, this is satisfied by the three element structure $0, 1, 2$ with $S(0) = 1, S(1) = 2, S(2) = 0$ or represented graphically



Theories and Models III

- What if we consider the infinite theory T containing the observations about the natural numbers S^n for each $n > 1$

$$(\forall x)((\underbrace{S(\ldots S(x))}_{n \text{ times}} \neq x) \qquad (S^n)$$

- Taking $S(x) = x + 1$, T still has both the natural numbers $(0, 1, 2, \ldots)$ and the integers $\ldots, -2, -1, 0, 1, 2, \ldots$ as models (and even the reals).

Theories and Models IV

- What if we create T' by adding to T the observation that there is a unique element (0) that isn't a successor?

$$(\exists! y)(\forall x)(S(x) \neq y) \tag{Zero}$$

- However, T' still has multiple (non-isomorphic[1]) models, e.g., both the natural numbers and the natural numbers with a copy of the integers tacked on after.

$$0, 1, 2, 3, \ldots \quad \ldots -2^*, -1^*, 0^*, 1^*, 2^*, 3^*, \ldots$$

[1] See appendix

Theories and Models V

- We might try and make our language more expressive by adding more relations $<$ and axioms involving this, but it turns out[2] that ..

- No matter what first order axioms we take our 'theory of the numbers' to consist of, there will be multiple distinct structures which satisfy this theory.

[2] by the Löwenheim–Skolem Theorem

Section 3

Putnam’s Challenge(s)

A Mathematical Realist Intuition

Intuition

We can reference the natural numbers not merely some alternate structure.

At least we seem to be able to reference them up to isomorphism,

- i.e., we can reference the structure of the natural numbers but perhaps not know if 0 is identified with a particular set or some other object.
- Going forward, I will sometimes omit the ‘up to isomorphism’ caveat.

Worry About Determinate Number Theory

- Putnam: It must be our theory of the numbers that secures the right reference, i.e., the acceptable references must make (most of?) our theory true.

- Model Theoretic Argument: Our first order theory can't rule out non-standard models as the reference of 'the natural numbers'.

Note About Importance

- Note: this worry doesn't just challenge definite reference (up to isomorphism) for our talk of the natural numbers. It also challenges

Intuition (Number Theoretic Truthvalue Realism)

Every number theoretic[3] statement is either true or false (whether we can prove it or not.)

- For without securing reference how do we secure truthvalues?

[3] That is a first order statement about the natural numbers expressed in terms of $0, +, *, <$.

Gödel Incompleteness

Theorem (Gödel's First Incompleteness Theorem)

Any first order theory T extending basic number theory[4] *whose axioms are algorithmically listable fails to prove or disprove some formula.*

- Equivalently[5] there is a sentence ϕ such that T has some models which make ϕ true and others which make ϕ false.
- So our first order theories can't secure truth-values for all number theoretic claims.
- However, determinate reference to the natural numbers would suffice to pin down number theoretic truth.

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[4] Such as Q or Peano Arithmetic.

[5] By the completeness theorem.

Analogous Challenge to Scientific Realism

- Putnam raises a similar worry about our ability to refer to physical objects and grasp scientific conceps in a way that makes it possible for an ideal scientific theory to be wrong.
  - Any consistent first order scientific theory can be interpreted as speaking truly about (some of) the sets.
  - So why don’t we count as speaking truely of this model?

A Common Combination of Views

It's currently common to:

- Take Putnam's argument as a serious challenge to our ability to reference structures like the natural numbers.
- Accept unproblematic realism about concrete objects, physical laws/possibility and objective probability.

But I will argue that this combination of views is untenable.

Section 4

Using Induction to Rule out Nonstandard Models?

Second Order Logic

- Second order logic extends first order by adding quantifiers $(\exists X), (\forall X)$ ranging over *all* subsets of a given domain.

- It's well known that if we can determinately grasp second order quantifiers then our acceptance of the Second Order Induction Axiom

$$(\forall X)\,[(X(0) \wedge (\forall n)(X(n) \rightarrow X(S(n)))) \rightarrow (\forall m)X(m)] \quad (I_2)$$

suffices to rule out all nonstandard models of first order number theory.

Second Order Categoricity

Theorem

Basic first order number theory plus the Second Order Induction Axiom

$$(\forall X)\,[(X(0) \wedge (\forall n)(X(n) \rightarrow X(S(n)))) \rightarrow (\forall m)X(m)] \qquad (I_2)$$

is satisfied only by the standard model of the natural numbers

- Definition: X is **counterinductive** within a structure/model if (restricting our quantifiers to that structure) $X(0)$ and $(\forall n)(X(n) \rightarrow X(S(n))$ but not $(\forall m)(X(m))$
- Second order induction is just the claim that no subset of the natural numbers is counterinductive.

Proof Idea

- Every model of basic first order number theory[6] has a standard initial segment. Such a segment is called an ω sequence.
    - Axioms tell us there is a first number 0 and every number has a successor giving us a standard initial segment $0, 1, 2, 3, \ldots$
- If *M* is non-standard, the standard initial segment $\omega \neq M$ is counterinductive. So *M* fails Second Order Induction.

[6] Again this means Q or Peano Arithmetic

Counterinductive Picture

Example (Putative Model of Natural Numbers)

0,1,2, … …-2*, -1*, 0*, 1*, 2* …

if X is in red in: $0, 1, 2, \ldots \; \ldots, -2^*, -1^*, 0^*, 1^*, 2^*, \ldots$

X **is not** counterinductive.

if Y is in blue in: $0, 1, 2, \ldots \; \ldots, -2^*, -1^*, 0^*, 1^*, 2^*, \ldots$

Y **is not** counterinductive.

if Z is in green in: $0, 1, 2, \ldots \; \ldots, -2^*, -1^*, 0^*, 1^*, 2^*, \ldots$

Z **is** counterinductive
(so rules out the above model)

Why This Doesn’t Answer Putnam

- Thus, if we give second order quantification its intended meaning and demand the numbers satisfy:
  - Basic first order number theory
  - The second order induction axiom

  then we must give the natural numbers a standard interpretation.
- But our ability refer to ‘all the subsets’ can seem no less mysterious than our ability to mean the standard model of the natural numbers.

Why First Order Induction Isn't Enough

- Our basic first order number theory can include the following induction induction schema

$$(\phi(0) \wedge (\forall n)(X(n) \to \phi(S(n)))) \to (\forall m)\phi(m) \qquad (I_1)$$

- But nonstandard models can satisfy all instances of this schema.
  - In a non-standard model no number theoretic ϕ picks out ω.

Tantalizing Idea

- We expect induction to hold for all formulas ϕ, including ones that are not in the language of pure number theory

$$(\phi(0) \wedge (\forall n)(\phi(n) \rightarrow \phi(S(n)))) \rightarrow (\forall m)\phi(m) \qquad (I_1)$$

- If we let ϕ use physical vocabulary maybe we can pick out ω (a counterinductive collection) in non-standard models.
- Remember, we assume that reference for physical vocabulary is fixed unproblematically.

Section 5

My Proposal

The Big Picture

I'll argue that, given determinate reference to physical objects and possibility, our acceptance of the following principle suffices to rule out nonstandard models of number theory:

Principle

It's physically necessary that the natural numbers satisfy

- the first order axioms of number theory and
- induction on the property 'there is an nth coinflip and it comes up heads'.

Counting Physical Events

Observation (Counting Events)

We can unproblematically count events[7] with numbers e.g., we can talk about $\mathrm{FLPCNT}(x, n)$ where $\mathrm{FLPCNT}(x, 0)$ holds if x is the first coinflip, $\mathrm{FLPCNT}(x, 1)$ if x is the second coinflip etc...

- At least when those events occur (temporarily) in a discrete sequence[8], i.e., .
  - There is a first event.
  - There is a well defined next event after every event $\forall y \exists x (y <_t x) \wedge \forall z \neg (y <_t z <_t x)$

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[7] See appendix for a more formal treatment of this

[8] We can restrict ourselves to events with a well defined temporal order in Relativity, i.e., timelike separation.

Induction applies to nonmathematical properties

- We can write formulas like $\phi(n) = (\exists x)\text{FLPCNT}(x, n)$ (There is an n-th coinflip).
- Note: the collection of numbers picked out by formulas using physical vocabulary may differ from that picked out by any purely mathematical vocabulary.

Principle

Induction holds for formulas using physical vocabulary. This is (taking quantifiers to be restricted to the 'natural numbers'

$$\phi(0) \wedge (\forall n)\,[\phi(n) \implies \phi(n+1)] \implies (\forall n)\phi(n)$$

How this can rule out some nonstandard models I

- We believe: if there is a first coinflip, *and* every flip is followed by a next flip, then there is an *n*-th flip for every natural number *n*.
- If there are finitely many coinflips, this doesn't rule out any putative models of the natural numbers.
- If there are infinitely many coinflips (ordered as below) this rules out non-standard models longer than the structure of ticks.

$$\underbrace{0, 1, 2, \ldots (\ldots -1^*, 0^*, 1^* \ldots)}_{\{n | (\exists x) \mathrm{CLKCNT}(x,n)\}} \ldots (\ldots -2^{**}, -1^{**}, 0^{**}, 1^{**}, 2^{**} \ldots)$$

How this can rule out some nonstandard models II

- If some describable kind of events (e.g., the coin flips) form an ω sequence, this will rule out all nonstandard models.
  - For, as we saw above, all non-standard models contain an initial ω sequence, plus some extra stuff.
- But: the actual world might not contain such an ω sequence

Key Idea #2

Solution:

- Appeal to our belief that the Inductive Principle above is physically/metaphysically necessary.

- If there's a possible world where a formula like ϕ picks out an ω sequence, this precludes nonstandard interpretations of 'the numbers' provided (as per physical realism) we give the usual meanings to:

  - 'physically necessary'/'metaphysically necessary'

  - 'before' and other physical vocabularly in ϕ.

On the Possibility of an ω Sequence

- Metaphysical possibility is easy.
  - Surely it's metaphysically possible for clock ticks to be an ω sequence.
- Physical possibility is harder.
  - Maybe it's not even physically possible for time to be an ω sequence (if infinite it must be non-standard).
  - But it's surely physically possible for time to be infinite.
  - We just need some way of selecting the standard initial segment.

One Last Little Trick I

Intuition

Given a discrete sequence of objectively random (and suitably independent) ‘coinflips’, e.g. quantum spin measurements, it’s possible for any collection of them to turn up heads.

- Physics tells us (in the right circumstances) these measurements are probabilistically independent so surely past outcomes can’t physically necessitate the current outcome.
- Even if you don’t believe in objective randomness, it’s plausible that the initial conditions are flexible enough to still guarantee this result

One Last Little Trick II

So if there can be some infinite discrete series of coin tosses (e.g., with this structure under $<_t$ 'before')

$$e_0, e_1, e_2 \ldots \ldots e_{-2*}, e_{-1*}, e_{0*}, e_{1*}, e_{2*} \ldots$$

Then plausibly all logically/combinatorially possible combinations of outcomes are physically possible, e.g.,

$$w_1 : H, H, T \ldots \ldots T, T, T, H, H \ldots$$

$$w_\omega : H, H, H \ldots \ldots T, T, T, T, T \ldots$$

including one world, w_ω, where the coinflips coming up heads form an ω sequence under $<_t$ and thus let us pick out a standard initial segment of the numbers (as discussed above).

Upshot

- Suppose that we can secure standard realist meanings for physical vocabulary like ‘coinflip’, ‘before’ and physical/metaphysical possibility talk.
- Then we can uniquely describe the structure of the natural numbers by saying:

Principle

It’s physically necessary that the natural numbers satisfy

- the first order axioms of number theory and
- induction on the property ‘there is an nth coinflip and it comes up heads’.

Section 6

Conclusion

Conclusion I

I have argued that a popular combination of views is untenable.

We shouldn't simultaneously

- take model theoretic worries seriously for number theory
- accept unapologetic realism about physical object and possibility talk

Conclusion II

For if we can secure realist reference to physical objects and possibility then we can rule out nonstandard models of number theory by saying:

Principle

It's physically necessary that the natural numbers satisfy

- the first order axioms of number theory and
- induction on the property 'there is an nth coinflip and it comes up heads'.

A Closing Note

A closing note

- It would be strange if our ability to grasp the natural number structure depended on our previously grasping a notion of physical (or metaphysical) possibility.
- So, I think that another - rather different- style of answer to Putnam's challenge must also be possible.
- See my blog entry `https://philosophyinprogress.blogspot.co.il/2017/10/access-to-reference-magnets-bitter-pill.html` for details on this.

Section 7

Appendix

*Note About Isomorphism

- Two structures M_1 and M_2 are isomorphic if they have the same structure but may differ in what objects are used.

- Somewhat more formally, a domain D considered under some relations $R_1, ..., R_n$ on that domain is isomorphic to a domain D' considered under relations $R'_1, ..., R'_n$ on that domain iff a function f can map D to D' such that:
  - f is one to one $(\forall x)(\forall y)(f(x) = f(y) \rightarrow x = y)$
  - f is onto $(\forall y))D'(y) \rightarrow (\exists x)f(x) = y)$
  - for each R_i, we have $(\forall x_1)...(\forall x_m)[R_i(x_1,,,x_m) \leftrightarrow R'_i(f(x_1),...f(x_m))]$
- e.g., the structures $0, 1, 2, \ldots$ and $-1, 0, 1, 2, \ldots$ under $<$ are isomorphic (n in the first structure plays the same role as $n - 1$ in the second).

Details About Counting Events

Let

- flip(x) denote 'x is a coinflip'
- $FLPCNT(n, x)$ denote x is the n-th coinflip
- $H(x)$ denote that coinflip x has the heads outcome
- $x <_t y$ denote that the coinflip x occurs temporally prior to coinflip y.

Then, I take the fact that $\Box_p$ (COUNTING RULES) to be somthing like an analytic truth, where COUNTING RULES is the conjunction of the following claims.

Definition of COUNTING RULES part 1

- An object x is the 0th coinflip, i.e., $FLPCNT(0,x)$ iff x is a coinflip and all other coinflips happen after x.
  $(\forall x)[FLPCNT(0,x) \leftrightarrow \text{flip}(x) \wedge (\forall y)(\text{flip}(y) \rightarrow x <_t y \vee x = y))]$
- If x is the nth coinflip, then y is the $S(n)th$ coinflip iff y occurs after x and no other coinflip occurs between x and y. That is,

$$(\forall n,x,y)\Big(FLPCNT(n,x) \rightarrow \Big[(FLPCNT(S(n),y) \leftrightarrow \text{flip}(y) \wedge x <_t y \wedge (\forall z)\neg(\text{flip}(z) \wedge x <_t z \wedge z <_t y)\Big]\Big)$$

Definition of COUNTING RULES part 2

- Only coinflips can be the *n*th coinflip, i.e., $(\forall x)(\exists n)(FLPCNT(n, x) \rightarrow flip(x))$

- No two distinct numbers correspond to the same coin flip. $(\forall n)(\forall m)[FLPCNT(n, x) \wedge FLPCNT(m, x) \rightarrow m = n]$

How this prevents nonstandard interpretations of $\mathbb{N}$ I

Consider the interpreter's predicament when interpreting 'countflip' in the world where there is a discrete sequence of coin flips and only the initial ω sequence comes up heads.

- The principles governing *FLPCNT* tell us that 0 has to be assigned to the temporally first coinflip in *w*, 1 to the next, and so on for all the objects in the standard initial segment of the nonstandard model.
- Thus if $\phi(n) \iff (\exists x)\,[FLPCNT(x, n) \wedge H(X)]$
- So we have ϕ applying to (at least) the standard initial segment of our nonstandad model.

How this prevents nonstandard interpretations of $\mathbb{N}$ II

- To make the induction axiom come out true we'd have to find some objects *n* to relate by *FLPCNT* to additional numbers to either
  - make the antecedent that ϕ applies to the sucessor of everything it applies to false,
  - or to make the conclusion that it applies to all 'numbers' true).
- But COUNTING RULES imply *FLPCNT* relates each number to a different coinflip.
- But we have already 'used up' all the ω sequence of successive coinflips that actually turned out heads by pairing them with the standard initial segment of our nonstandard model,

Upshot

We can write down a first order theory, which we expect to hold necessarily,

- basic first order axioms of number theory, e.g., the Peano Axioms
- COUNTING RULES
- the sentence you get by instantiating the first order induction schema with $\phi(n) = (\exists x)(FLPCNT(n, x) \wedge head(x))$

but which no nonstandard interpretation of 'number' which interprets 'heads', 'coinflip' 'before' etc standardly at w_ω can make come out true at w_ω

In the existing lit: McGee

McGee argues that that we expect instances of the induction schema to remain true in all ‘logic preserving’ expansions of our language.

- If we met a god and adopted their term ‘smee’ which applied only to the standard initial segment of our non-standard model, induction would fail.
- But what can the god do to secure reference to a collection of would-be numbers in a way we cannot?
- Our ability to grasp McGee’s space of ‘logically possible logic preserving language changes’ is no less mysterious than our ability to second order logic (or the intended structure of the natural numbers).

In the existing lit: Hartry Field

Hartry Field (rather ambivalently) proposes an alternative account:

- If seconds since the epoch forms a genuine ω sequence, i.e., has the intended structure of the natural numbers under $<$.
- and we believe this to be true.
- then we can rule out non-standard interpretations.

Problem for Field's proposal

- Time could be finite or non-standard, e.g., maybe there are times *after* an infinite duration.

- Even if time has the right structure, this is (at best) a contingent hypothesis and not sufficiently analytic/central to our use to rule out alternative models.

    - Compare: If I believed that the number of gumballs in the jar is 70, *this* belief presumably wouldn't commit a mischievous interpreter to interpret the concept 'natural number' so this statement come out true.