THE RESIDUAL ACCESS PROBLEM

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Abstract. A range of current truth-value realist philosophies of mathematics allow one to reduce the Benacerraf Problem to a problem of explaining our accuracy about which mathematical practices are coherent – in a sense which can be cashed out in terms of logical possibility. However, our ability to recognize these facts about logical possibility poses its own access problem. I'll propose a solution to this residual access problem for logical possibility and suggest that accepting powerful and correct general principles for reasoning about logical possibility can be the most efficient way to predict and explain the behavior of concrete objects.

1. Introduction

It's appealing to think that there are right answers to all arithmetical questions and many questions in other mathematical domains – regardless of whether our proof practices will ever let us discover these answers. However, accepting such truth-value realism raises an explanatory problem. What can explain our true beliefs about mathematics, i.e., the match between human psychology and objective mathematical facts?

The history of past attempts to respond to this problem can make it appear that no adequate explanation of human accuracy about mathematics is conceivable. Accordingly, accepting truth-value realism about mathematics can seem to require positing an 'extra' inexplicable coincidence (i.e., one which could be avoided by adopting a less realist philosophy of mathematics). This objection is sometimes called the access problem.

A range of current truth-value realist philosophies of mathematics (within what I will call the structuralist consensus) allow one to reduce the this problem of explaining our accuracy about mathematics to a problem of explaining our accuracy about which mathematical practices are coherent – in a sense which I will cash out using a logical possibility operator. However, this leaves us with a residual access problem concerning logical possibility.

In this paper I will propose a solution to this residual access problem (by expanding on some ideas proposed in [2]). That paper sketches an explanation for our ability to recognize the logical possibility or impossibility of various first order logical states of affairs along the following lines. We get initial knowledge of logical possibility via the fact that what is actual must be possible. We can then expand this knowledge via something like inductive generalization – as accepting powerful and correct general principles for reasoning about logical possibility can be the most efficient way to predict and explain the behavior of concrete objects [we are then able to expand our knowledge of logical possibility.

However, accounting for our knowledge of which first order scenarios are logically possible is not enough to solve the access problem. The problem is that philosophers in the structuralist consensus think that mathematical posits employ non-first order logical vocabulary. For, by familiar Gödelian considerations, some such vocabulary is necessary to categorically describe paradigmatic mathematical structures like the natural numbers[1]. So to banish access worries we need to account for our ability to recognize the logical possibility of mathematical posits made using some logical vocabulary powerful enough to categorically describe mathematical structures like the natural numbers (e.g., posits in the language of second order logic).

[1]No algorithmically listable first order axioms can describe the natural numbers sufficiently to rule out varying interpretations which give different answers to some question in the language of arithmetic.

It's not clear how the story proposed in [2] could explain such knowledge. While the actual to possible move provides us with initial knowledge that certain first order states of affairs are logically possible, it is less clear how it could give us any knowledge of the logical possibility of states of affairs described using second order quantification. For example, insofar as we can't see or touch or taste etc. second order objects like collections (as opposed to the concrete objects which can figure in first order reasoning), these objects seem to raise all the same access worries as mathematical objects, and our accuracy about them cannot be presumed. If we don't presume this kind of starting point (knowledge that certain second order states of affairs are actual), then it's hard to see how the above mechanisms could account for our knowledge of general laws governing which second order states of affairs are logically possible.

In this paper I will propose an account of our knowledge of the logical possibility of a certain kind of claims - claims in a language with sufficient power to categorically describe pure mathematical structures like the natural numbers. In section 3 I will introduce a notion of conditional logical possibility, and note how it can do all the work which second order logic traditionally does in categorically describing mathematical structures. In section 4 I will deploy a version of the mechanisms of correction proposed in [2] and sketched above, to explain knowledge of conditional logical possibility claims. In section 5 I will address some objections to the resulting proposal.

2. Setting up the problem

Following [10] it is popular to think about access worries for (mathematical) realists as arising from a challenge for the realist to "explain how our beliefs about [mathematical objects] can so well reflect the facts about them" in some internally coherent fashion. More specifically, Field demands that we explain the truth of 'reliably, if mathematicians believe that ϕ then ϕ', for various mathematical statements ϕ. And he notes that, "[I]f it appears in principle impossible to explain this, then that tends to undermine ... belief in mathematical entities, despite whatever reason we might have for believing in them." It has been very popular to construe access worries about morals, metaphysical possibility, mathematical truth-value realism, aesthetics etc. analogously.

On this view, access worries provide us ceterus paribus reason for rejecting a theory (in this case realism about mathematical objects), by appealing to a kind of cumulative gestalt impression that no adequate answer to this explanatory demand seems possible. The mathematical realist is presented with a challenge to explain how, if their doctrine is true, human accuracy about mathematics could be anything but a miracle or a mystery (given e.g., our inability to see or touch or taste or otherwise causally contact mathematical objects). Insofar as no satisfying answer to this challenge seems possible, the realist appears committed to positing extra coincidences, which could be avoided by adopting a less realist philosophy of mathematics[2].

Now [2] argues that we can answer this access problem for truth-value realists about mathematics by providing a simplified example of a mechanism which could explain human accuracy about realist mathematics[3]. It

[2]See REDACTED on the desirability of sticking to this intutive conception of access worries in terms of apparent commitment to some extra coincidence explanation in blocking trivalizing solutions to Field's access worry (which explain one apparently and coincidental match between human psychology and supposed objective mathematical/moral facts by appealing to another) a la [5] and [7].

[3]One might wonder how giving any such *simplified* example of an explanation (which thereby likely fails to match actual human history in various ways) could suffice to answer access worries. But, one can think about access worries as presenting a 'how possibly' question ('how can we have gotten significant reliability about realist mathematics?') wherein we are challenged to reconcile the apparent reality of some state of affairs with certain obstacles (consciously recognized or not) which make this state of affairs seem impossible. Call these obstacles blocking conditions. And (c.f., [3] and [19]) one can answer a 'how possibly' question by giving an example of an explanation for the relevant

then suggests that a suitable sample explanation can be developed along the following lines. First one notes that many truth-value realist philosophies of mathematics (those within the 'structuralist consensus') imply that one can reliably form true mathematical beliefs by adopting nearly any logically coherent bundle of pure mathematical axioms. Accordingly they let one reduce the classic access problem (of explaining our accuracy about mathematics) to a problem of explaining our accuracy about which mathematical practices are coherent.

The second step is to explain how creatures like us (with all the blocking conditions that seem to make access to truthvalue realist mathematics seem mysterious[4]) could have gotten knowledge of suitable logical possibility facts – that is, knowledge of the logical possibility of claims in a language with sufficient expressive power to categorically describe paradigmatic mathematical structures like the natural numbers. That will be my task in this paper.

3. Modalizing our Mathematical Concepts

My first step will be to introduce a notion of logical possibility, conditional logical possibility, which allows certain facts to be held fixed. In this section I will motivate and introduce this key concept and note that it can do all the

state of affairs which is compatible with all the obstacles – even if there is little reason to believe that this explanation is true. Indeed, one can often best answer such how possibly questions by giving a *simplified* explanation which gets certain details of known history wrong – provided that all blocking conditions are accommodated and the core explanatory mechanism being demonstrated is sufficiently adaptable.

[4]I take the relevant blocking conditions to include abstractness and metaphysical necessity of mathematical facts, our lack of causal contact with mathematical objects, and our knowledge of mathematical theories in advance of any any scientific applications. Of course whether such a simplified explanation succeeds in answering a 'how possibly' question depends on the specific blocking conditions one takes to be part of that question. So someone could always re-raise the access problem for truth-value realism about mathematics by citing features of our actual phenomenology/biology/history which recreate the appearance that no explanation is possible. But I'm not aware of any way that my story differs from reality would seem to create such an impression.

work of second order logic in articulating our conceptions of mathematical structures.

3.1. **Conditional Logical Possibility.** First note that (as discussed in [2]) we have an intuitive notion of logical possibility, which applies to claims like $(\exists x)(red(x) \wedge round(x))$ and makes sentences like the following come out true.

- It is logically possible that $(\exists x)(red(x) \wedge round(x))$
- It is not logically possible that $(\exists x)(red(x) \wedge \neg red(x))$
- It is logically necessary that $(\forall x)(red(x)) \rightarrow \neg(\exists x)(\neg red(x))$.

Philosophers representing a range of different views of mathematics have made use of this notion[5] and are comfortable applying it to non-first order sentences. If you are skeptical that there is such a notion, note that it is definable in terms of the even more common notion of validity (it's logically possible that ϕ iff it's not logically necessary that $\neg\phi$ iff its not valid to infer $\neg\phi$ from empty premises).

Like Hartry Field[11], I think it is attractive to take this notion of logical coherence to be a primitive concept, that does not need to be cashed out by appeal to any (conceptually or metaphysically) prior facts about set theoretic models or possible worlds[6]. Thus, we don't face problems of epistemic

[5]See the discussion of the corresponding notion of consequence in [9],[20] and [16].

[6]Admittedly, there's now a fruitful tradition of identifying logical possibility/validity with having a set theoretic model for various mathematical purposes. However as [11] 2.3 and Etchemendy [8] note, this is better understood as a happy outcome of substantive mathematical and philosophical insight produced by the completeness theorem regarding the special case of first order logic, than any argument that our concept of logical possibility must be grasped via appeal to claims about set existence or should be grounded in such claims.

For, as Field compellingly argues in [11], it's core to our conception of this logical possibility simpliciter that what's actual is logically possible. But if we think about logical possibility in terms of set theoretic models, then the actual world is strictly larger than the domain any set theoretic model (e.g. because it contains all the sets), so it's not prima facie clear why we should assume that what can't be satisfied in any set theoretic model isn't actually true. Thus we seem to antecedently grip a notion of logical possibility (interdefinable with validity) on which its an open question whether every logically possible state of affairs has a set theoretic model.

or metaphysical regress if we want to explain our access to mathematical objects using our access to facts about logical possibility.

Now I propose that this notion of logical possibility allows for a natural generalization – to what I will call the notion of 'conditional logical possibility'. For consider a sentence like, "Given what cats and baskets there are, it is logically impossible that each cat slept in a distinct basket." There's an intuitive reading on which this sentence will be true if and only if there are more cats than baskets[7]. This reading employs a notion of logical possibility *holding certain facts fixed* (in this case, structural facts about what cats and baskets there are[8]).

I will use a conditional logical possibility operator $\Diamond$ (and the corresponding $\Box$), which takes a sentence ϕ and a (potentially empty) list of relations $R_1 ... R_n$ and produces a sentence $\Diamond_{R_1 ... R_n}\phi$ which says that it is logically possible for ϕ to be true, given how the relations $R_1 ... R_n$ apply. Thus, for example, the claim, 'Given what cats and baskets there are, it is logically impossible that each cat slept in a distinct basket' becomes:

It is indeed *currently* possible for mathematicians talking about *first order logical sentences* to replace talk of logical possibility with talk of set theoretic models, by way of the completeness theorem. But the completeness theorem is best seen as providing a sandwiching argument that three antecedently distinct concepts turn out to be coextensive when applied to first order logical sentences. For intuitive everything that has a set theoretic model is logically possible, and everything that's logically possible is syntactically consistent in the sense of not letting one derive contradiction via applying the rules of FOL. So when the completeness theorem shows that a first order logical sentence satisfies the strong condition (having a model) iff it satisfies the weaker condition (permitting a proof of contradiction in FOL), this tells us that we can harmlessly replace talk about logical possibility with talk about having a model in the special case where we are talking about logical possibility.

[7]Admittedly, there's another reading of this sentence on which it expresses a necessary falsehood. However, this is not the reading I have in mind.

[8]Hellman's own use of logical possibility given the material facts commits him to the coherence of something very much like this notion.

(CATS)

$$\Box_{\text{cat,basket}}\neg\Big((\forall x)\Big[\text{cat}(x)\rightarrow(\exists y)\,(\text{basket}(y)\wedge\text{sleptIn}(x,y))\Big]\wedge (\forall z)(\forall w)(\forall w')\Big[\text{basket}(z)\wedge\text{cat}(w)\wedge\text{cat}(w')\wedge \text{sleptIn}(w,z)\wedge\text{sleptIn}(w',z)\rightarrow w=w'\Big]\Big)$$

Remember that when evaluating logical possibility we consider all possibilities for the relations mentioned in the statement under consideration, whether we can describe them or not. This is analogous to requiring second order quantifiers to range over all possible collections.[alt somethign like: georgie said prev was confusing: When evaluating claims about logical possibility, I think we consider all possible ways that some predicate could apply, whether uniquely describable or not.] And this is the key fact which will let us rewrite second order logical descriptions of mathematical structures with equivalent descriptions in terms of conditional logical possibility[9].

[9]Also note that one can perhaps get correct truth conditions by thinking of $\Diamond_{R_1\ldots R_n}\phi$ claims as holding fixed the *particular objects* in the extension of the relations $R_1 ... R_n$ – and then asking *de re*, of these objects, whether one could supplement them with other objects (and choose extensions for all other relations) so as to make ϕ true. However, there seems to be a primitive notion of preserving the *structural facts* about how some relations apply, which does not depend on our understanding any such controversial *de re* claims. For example, in the case of CATS these will be scenarios which agree with the actual world on: the number of objects satisfying cat(), the number satisfying basket() and the number of things in the extension of both cat() and basket(). However, preserving the structural facts does not require preserving facts about identity (or supposing that the relevant 'cross logical-possibility counterparthood' facts are well defined).
If, say, one cat died and an additional kitten was born, these structural facts would remain unaltered. Crudely, we might gesture at the idea of preserving the structural facts by saying that two scenarios have the same structural facts about the relations $R_1, \ldots, R_n$ if the objects satisfying some R_i in the first scenario (more precisely those x such that $\exists y_1, \ldots, y_k, y_{k+2}, \ldots, y_m R_i(y_1, \ldots, y_k, x, y_{k+2}, \ldots, y_m)$ for some i, k and m.) are 'isomorphic' (under $R_1, \ldots, R_n$) to the objects satisfying some R_i in the second scenario. Note that this is not intended to be a definition of the concept, only an attempt to point at the correct primitive notion, as the very notion of isomorphism would be defined in terms of logically possible mappings.

For example, we can express the induction axiom for number theory (which is usually expressed in second order logic) as follows.

> **Induction Axiom**: if some property applies to 0 [10] and to the successor of every number it applies to, then it applies to all the numbers.

- **Induct**: 'It is logically necessary, given how number and successor apply, that if 0 is happy and the successor of every happy number is happy then every number is happy.' i.e., $\Box_{\mathbb{N},S}$(if 0 is happy and the successor of every happy number is happy then every number is happy).

Thus, we can write a sentence $PA_\Diamond$, which categorically describes the natural numbers [11] and hence ensures that for every sentence of number theory ϕ, either ϕ or $\neg\phi$ is a logically necessary consequence of $PA_\Diamond$[12]. Note that while PA_2 is traditionally expressed in terms of the relations number, successor etc. we can substitute any other relations of the same arity (happy, loves etc.). As logical possibility ignores any particular features of relations (unless conditioned on) the truth value will be unaffected[13].

We can also make nested claims about logical possibility. Note that in a nested claim with the form $\Diamond\Box_R\psi$, the subscript freezes the facts about how the relation R applies in the scenario being considered, which may *not* be the state of affairs in the actual world. So, for example, POSSIBLY

[10] By '0' I mean the unique number which is not the successor of any number, and I take the rest of the sentence to be spelled out using this definite description in standard Russellian fashion.

[11] Essentially, $PA_\Diamond$ conjoins the finitely many axioms of first order PA with the statement of induction in terms of logical possibility (Induct) above. The only other modification needed is to replace mathematical vocabulary with non-mathematical vocabulary ('number', 'successor' etc.) with non-mathematical vocabulary of the same arity, as per Putnam's suggestion above.

[12] That is $\Box(PA_\Diamond \rightarrow \phi)$ or $\Box(PA_\Diamond \rightarrow \neg\phi)$

[13] So, for example, the sentence $\Diamond(\exists x)(\exists y)(\text{Dog}(x) \wedge \text{Cat}(y) \wedge \neg x = y)$ and the sentence $\Diamond(\exists x)(\exists y)(\text{Dog}(x) \wedge \text{Lemur}(y) \wedge \neg x = y)$ always have the same truth value.

CATS (below) expresses a metaphysically necessary truth. For, whatever the actual world is like, it will always be logically possible for there to be, say, 3 cats and 2 baskets, and this scenario is one in which it is logically necessary (holding fixed what cats and baskets there are) that: if each cat slept in a basket then multiple cats slept in the same basket. So it is metaphysically necessary that POSSIBLY CATS.

(POSSIBLY CATS)

$$\Diamond\Box_{\text{cat},\text{basket}}\neg\Big((\forall x)\Big[\text{cat}(x)\to(\exists y)\left(\text{basket}(y)\wedge\text{sleptIn}(x,y)\right)\Big]\wedge (\forall z)(\forall w)(\forall w')\Big[\text{basket}(z)\wedge\text{cat}(w)\wedge\text{cat}(w')\wedge \text{sleptIn}(w,z)\wedge\text{sleptIn}(w',z)\to w=w'\Big]\Big)$$

In [1] Berry argues more formally for the claim that this notion of conditional logical possibility can replace second order quantification generally. Note that we will refer to the language which is built up using relations, first order connectives and the conditional logical possibility operators $\Diamond_{R_1,\ldots,R_n}$ and $\Box_{R_1,\ldots,R_n}$ (where the these two operators can only be applied to sentences, so there is no quantifying in) the language of logical possibility.

4. Knowledge of Conditional Logical Possibility

Now let us turn to the residual access problem for logical possibility, which arises if we think about pure mathematical posits in the way I have advocated in the previous section. How could creatures like us reliably recognize the logical coherence of mathematical hypotheses (formulated in the language of logical possibility above)[14]?

[14]Note that this ability is different from the ability to unerringly decide whether an *arbitrary* mathematical hypotheses is coherent – something we couldn't in principle do for familiar Gödelian reasons. The reliability I'm invoking here is that -although we often

In this section, I will propose a story about how creatures initially equipped with only the kind of non-mathematical faculties philosophers pressing an access worry are willing to grant (e.g., observation, first order logical deduction, scientific induction) might have developed good methods of reasoning about logical possibility. To appreciate the scope of what needs to be explained, note that merely using the correct introduction and elimination rules for first order logic does not allow one to recognize the positive fact that a scenario is logically coherent. For example, first order introduction and elimination rules don't allow one to recognize that it would be logically coherent for there to be two distinct things $(\exists x)(\exists y)\neg x = y$[15].

As inquirers, we attempt to predict and explain the behavior of concrete objects. There are more and less economical ways of doing so. When we are dealing with sufficiently diverse and plentiful collections of concrete objects, the most economical explanations for regularities may well appeal to a combination of general principles which constrain how any objects can be related by any relations, and specific physical or metaphysical laws whose application is restricted to certain particular kinds of objects or relations. I will suggest that, in this way, pressure to efficiently predict physical events in situations of evolutionary interest can help explain how creatures like us could have gotten a concept of logical possibility and then developed powerful methods of reasoning with it.

My story begins with the idea that our compositional language admits many different-looking representations whose falsehood is guaranteed by

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suspend judgment on such claims- in the cases where we do form judgments we tend to be correct (and the same holds in all close possible worlds).

[15]By Gödel's completeness theorem[14] it turn out that every logically incoherent first order scenario allows for a derivation of contradiction using the usual inference rules for first order logic. But this was a substantive result which it took real mathematics to prove, so not something the denizens of our story would or could assume. Furthermore, humans clearly don't infer that a scenario is logically coherent by checking all possible proofs (whose premises are true in the scenario) for a contradiction.

logic alone[16]. Thus many plans which we can verbally represent can be discarded as unrealizable purely on the grounds that they require something logically impossible. And there are practical benefits to be gained from being able to systematically recognize and focus our attention on those plans which are, at least, logically possible[17].

Accordingly I think it would be unsurprising if we eventually (either consciously, unconsciously or at the level of evolutionary selection) began to exploit the fact that certain linguistic patterns yield falsehoods no matter the content of the relations being represented – and acquired (something like) a notion of logical possibility including the following two principles (and the expectation that $\Diamond$ facts should follow elegant general laws).

- $\phi \rightarrow \Diamond_{R_1 \ldots R_n} \phi$
- $\Diamond_{R_1 \ldots R_n} \phi \leftrightarrow \Diamond_{R_1 \ldots R_n} \phi[S_1/S'_1 \ldots S_m/S'_m]$ where none of the S_i, S'_i are among the R_i

The first principle embodies the idea that we are talking about a notion of possibility, saying that everything actual is logically possible. The second principle embodies the idea that we are talking about possibility with respect to logical form alone, so that systematically replacing one relation with

[16]One might naturally wonder why scenarios which we can predict won't be realized are ever entertained at all. Wouldn't it be better if we couldn't even formulate the idea of such scenarios? Surely it would be, if we could filter out impossible scenarios with little cost. But this is plausibly forbidden by computational complexity considerations. For obvious reasons it is useful to consider conjunctions, disjunctions and negations of scenarios which means that any such filtering would be, at a minimum, required to do the equivalent of filer out all unsatisfiable boolean formulas. Yet, if $P \neq NP$ then by the Cook-Levin theorem [6] then no polynomial time algorithm can accomplish such filtering. These considerations render it very plausible that we would be inclined to consider a great many scenarios (such as unsatisfiable boolean combinations) many of which we may later be able identify as belonging to particular class of scenarios which will never be realized. Thanks to REDACTED for making this cute point.

[17]Note that even though creatures with first order logic will already be disposed to reject plans when they derive a contradiction from them, there are further benefits to be gained from having a positive theory (e.g., being able to infer that one scenario is logically possible only if another one is, allows one to skip searching for a contradiction in the former scenario after seeing the later scenario realized).

another doesn't change logical possibility facts. Note that many natural variants on this initial conception of logical possibility would intuitively count as getting something else right (e.g., setting out to learn facts about physical, chemical, metaphysical, or psychological possibility) rather than getting logical possibility wrong.

I will now attempt to explain how creatures (with familiar human observational, scientific-inductive etc. faculties but no mathematical knowledge) could go from having this kind of minimal conception of logical possibility to having powerful methods of reasoning about logical possibility sufficient to capture all of contemporary mathematics (understood in a modal structuralist vein).

4.1. **Three Mechanisms of Correction.**

I will propose three key ways in which dealings with concrete objects can (directly or indirectly) help correct and accurately extend our methods of reasoning about logical possibility.

4.1.1. *From ϕ to $\Diamond\phi$ and $\Diamond_{R_1...R_n}\phi$ facts.*

Recognizing relationships between concrete objects can push us to accept some $\Diamond\phi$ statements. Imagine that you aren't sure whether the state of affairs described by some mathematical hypothesis involving relations P, Q, and R is logically possible. If I then point out that the relations of friendship, nephew-hood and having been in military service together apply in just this way to the royal family of Sweden, this will cause you to accept that the scenario in question is logically possible.

Similarly, recognizing actual relationships between concrete objects can create systematic pressure to accept particular claims about subscripted logical possibility. Just as what is actual is logically possible, what is actual is logically possible given any facts about the actual world [18]. Thus, for example, one can go from 'every dog loves some human' to '$\Diamond_{dog}$ every dog loves

[18] That is, for any collection of relations $R_1...R_n$ and state of affairs ϕ, if ϕ then it is logically possible that ϕ given the facts about how relations $R_1...R_n$.

some human'[19] or '$\Diamond_{human}$ every dog loves some human' or '$\Diamond_{dog,human,loves}$ every dog loves some human'. In this way recognition of actual relationships can also create pressure to accept certain $\Diamond_{R_1 \ldots R_n}\phi$ claims.

The advantages to be gained by recognizing useful physically possible scenarios can also create pressure to accept *general inference methods*[20] which allow one to recognize hitherto unrealized scenarios as logical possibilities. As a result, the benefits to be gained from recognizing physical possibility facts can push us towards methods of reasoning which allow us to arrive at the logical possibility of non-actual states of affairs. Similarly, it can also be useful to recognize what is possible while keeping certain relations fixed, and this can help explain our tendency to accept general inference methods which let one reliably derive true claims about conditional logical possibility.

For instance, consider someone who didn't accept (even finitary) choice as a valid inference method for logical possibility. That is, they weren't willing to infer from $(\forall x)(D(x) \to (\exists y)R(x,y))$ to $\Diamond_{R,D}(\forall x)(\forall y)(F(x,y) \to R(x,y)) \wedge (\forall x)(D(x) \to (\exists! y)F(x,y))$. Such an individual might know that the enemy has divided their army up into platoons (so $D(x)$ is true just if x is a platoon in their army) and know that every platoon had at least one soldier ($R(x,y)$ holds just if y is a soldier in platoon x) but yet be unsure if it was (even logically) possible for the enemy to select a single soldier in each platoon to be the platoon leader. Failing to recognize such a possibility would be disadvantageous, and the fact that in every circumstance where the question arose there was always such a choice relation would create pressure to accept such an inference procedure. Admittedly, this is a somewhat simple and contrived example but (as with mathematics) reasoning about

[19]Remember that this means considering a scenario which preserves the number of dogs in the actual world.

[20]I take these to include things like the use of inference schemas and various ways of manipulating mental and physical pictures.

logical possibility really comes into its own when we put multiple inferences together to reach more complex conclusions.

4.1.2. *From $\neg\phi$ facts to $\neg\Diamond\phi$ and $\neg\Diamond_{R_1 \dots R_n}\phi$ claims.* Even though the non-actual need not be non-possible, our need to elegantly explain regularities in the concrete world creates pressure to conclude certain states of affairs are logically impossible. Suppose, for example, that someone thought it was logically possible for 9 items to differ from one another in which of three properties they had, e.g., for 9 people to choose different combinations of sundae toppings from a sundae bar containing three toppings. This person would have to explain the striking law-like regularity that, regardless of the type of items and properties in question, we never wind up observing more than 8 such items. They might postulate new physical regularities to explain why apparently random processes of flipping three coins never generated the forbidden 9th possible outcome. However, this explanation (or some analogous one) would have to apply at every physical scale we can observe, from relationships between the tiniest particles to relationships between planets and stars (as well as to less concrete objects like poems and countries). A much more elegant explanation is that the unrealized outcome is logically impossible. Recognizing that the forbidden 9th outcome is forbidden in all possible domains is much more efficient than hypothesizing separate laws prohibiting it in each specific situation (and thus there is pressure to do so).

This mechanism also provides pressure to accept conditional logical possibility claims. For example, if we keep noticing that when there are 4 cats and 3 baskets it is never the case that each cat slept on a different basket, the most elegant explanation for this is that it would be logically impossible for each cat to have slept on a different basket.

Accordingly, we can think of facts about what's actual as simultaneously a useful source of data about what's logically possible, physically possible, chemically possible, etc. We try to efficiently predict what will happen by patching together laws with different levels of generality. Though (in principle) we always face a choice about whether to explain a given regularity in terms of logical necessity, physical law, metaphysical necessity or mere ceterus paribus regularity, patterns in our experience can still motivate attributing a noted regularity to logical necessity rather than physical law. For, as noted above, if a regularity holds as a matter of logical necessity, we should expect to see that all substitution instances of it (i.e., all sentences with the same logical structure) are true, whereas we would expect the opposite if some principle holds as a matter of merely metaphysical necessity or physical necessity. This is not to say that we always make the right judgment, but in the long term we face significant pressure to correct our mistakes.

4.1.3. *Approaching Reflective Equilibrium.* Finally, one should note that the pressures mentioned above don't exist in isolation. Rather the resulting beliefs (and inference methods) will be further corrected by interaction with one another. If one accepts the above story about how we could have gotten some initial 'data points' about logical possibility from our knowledge of the concrete world, one can then appeal to familiar processes of reflecting on our beliefs and recognizing when they conflict or cohere with one another to explain some further improvements in our accuracy[21] [22].

Once some methods of reasoning come to strike us as initially attractive via the two mechanisms above, we can arrive at new more powerful laws (just as we do in the sciences) by considering how they unify and explain these methods of reasoning. For example, in the literature on the search for new axioms in set theory it has often been argued that we can reliably add new axioms by choosing principles which unify and explain the mathematical beliefs which we already have[17]. As Gödel puts it, "There might exist axioms so abundant in their verifiable consequences, shedding so much light upon a whole field, and yielding such powerful methods for solving problems... that, no matter whether or not they are intrinsically necessary, they would have to be accepted at least in the same sense as any well-established physical theory"[15]. If this is true, then it also seems plausible that the creatures in our just so story might reliably expand an initial collection of good methods of reasoning about logical possibility in the same way. Moreover, when we make incorrect generalizations these can be corrected by coming into conflict with well-entrenched and concretely motivated general principles.

Finally, remember that that the kind of elegant generalization which we see in the sciences (and which I want to invoke) goes beyond simple inferences

[21]While the notion of reflective equilibrium is frequently invoked to provide justification for a theory, I also take it to be widely presumed that it's a reliable process when given some initial degree of accuracy. Even philosophers who critique the use of reflective equilibrium to address access worries (such as Justin Clarke-Doane[4]) tend to formulate their objection as kind of 'garbage in garbage out' worry. Such philosophers seem to accept the reliability of induction and reflective equilibrium *if some initial accuracy about mathematics/morals could be generated*, but argue that it's puzzling how we could get initial basis of accuracy about particular mathematical and moral facts for reflective equilibrium to operate on.

[22]This process of reflective equilibrium relies on our ability to make valid logical inferences (though not our ability to recognize logical possibility). However, as noted above, we can take initial proficiency with first order (classical) logical vocabulary for granted, since solving the access problem for truth-value realist philosophies of mathematics merely requires explaining the reliability of our mathematical beliefs *given* widely accepted background assumptions shared by truth-value realists and anti-realists alike.

like 'the sun rose every day for the past billion years, so it will rise tomorrow.' It can include the kind of, seemingly astonishing, leaps we see in the sciences like going from observations of points of light in the night sky to a whole model of how the planets are arranged.

5. Worries

I will now consider a family of objections to the story above. These objections arise from the following simple idea: we causally interact with relatively small collections of objects, but one needs to appeal to accuracy about the logical possibility of much larger (typically infinite) collections to explain our accuracy about mathematics. In this section I will address three worries which arise from this apparent gap.

5.1. **Scientific Induction Unreliable in Mathematics?** First, one might worry that scientific-induction-like generalization from cases (whether it be implemented consciously, unconsciously or evolutionary) is completely unreliable with regard to mathematics. If this were correct, it would certainly raise a problem for my proposal that dealings with small concrete collections could have pushed us to develop accurate general methods of reasoning about logical possibility.

However, there's strong independent reason to reject insinuations that generalization from cases is completely unreliable in mathematics. Mathematicians frequently use hunches developed from past experience, judgments of general plausibility or theoretical attractiveness and the results of computational searches [23] to guide their research. For example, belief that Fermat's last theorem was true *before* a proof found was motivated by consistent failure to find a counterexample. If we want to make sense of the

[23] Of course, they do not do this naively. If they already know that counterexamples would have to be huge they wouldn't change their judgments because no small counterexamples were found.

apparent success of this aspect of mathematical practice, we can't suppose that something about the nature of mathematics makes the kind of elegant generalization from cases we find in the sciences *completely unreliable* when applied to the mathematical realm[24].

5.2. **A Gap Between the Finite and the Infinite?** Next, one might worry that the story suggested above cannot explain the *degree* of mathematical knowledge we take ourselves to have. Specifically one might worry that my mechanisms of correction could explain human accuracy about logical possibility facts involving finite collections, but not accuracy about the logical possibility of even the smallest infinite collections, like the natural numbers[25].

To address this worry, I will make two points.

First, I claim that the physical world around us is (apparently) helpfully describable in terms of infinite spatial sections[26]. Consider, for example, the stretches of space along the path of an arrow, or the stretches of time during which the arrow is traveling[27].

Note that this claim that (in many ordinary contexts) it is useful to conceptualize/idealize the macroscopic physical world around us in terms

[24]Of course, this is not to say that mistakes in generalizing from the patterns of how relations can apply to (often concrete) non-mathematical objects to accurate general principles about logical possibility is infallible. For example, one might argue that Freiling's argument [13] against the Continuum Hypothesis illegitimately transfers intuitions about physical space to constraints 0on logical possibility and set theory.

[25]See Frege's [12] pg. 16 for a version of this objection. He suggests that different numbers are like different geological strata and that one cannot infer facts about one from the other.

[26]Note that my project of using experiences with non-mathematical objects (along with knowledge of first order logic, ability to do scientific induction etc.) to *solve the access problem* differs crucially from historical attempts to reconcile mathematical truth-value realism with some empiricist thesis requiring all non-logical knowledge to come from the senses (or some materialist thesis saying that all objects must be material, which would rule out the existence of such chunks of space).

[27]Note that I don't presume (or need to presume) that concrete reality forces any single unique such structure on us. As Penelope Maddy emphasizes in [18] science and philosophy of science may underdetermine what logico-mathematical structure to ascribe to a physical system.

of infinitely many spatial segments is compatible with the possibility that space is quantized or there aren't distinct objects called spatial segments or spatial points at all. Whether or not such 'quasi-physical' objects as spatial paths actually exist (or are physically fundamental if they do exist), it is clearly very useful for us to think about space in terms of them in many contexts.

Second, I claim that the mere practical usefulness of thinking in terms of certain structures is a somewhat reliable (though not infallible) sign of logical possibility. And this is all we need to explain how the usefulness of talking in terms of some non-concrete structures can be helpful in explaining our accuracy about logical possibility. This intuition is notably shared even by many philosophers who would deny that such usefulness is a good guide to existence. This aspect of my proposal resembles Quine's idea that scientific usefulness of mathematical structures is a guide to mathematical actuality, but I am instead making the much less controversial claim that the usefulness of abstract structures is a good guide to their logical possibility[28]

Additionally, even if you don't accept that we have access to any infinite physical collections, reasoning about how it would be logically possible for physical objects to be supplemented with an infinite collection of abstract objects can be very useful in stating elegant laws which predict and explain the behavior of physical objects. Consider the task of predicting what physical inscriptions of series of letters one will ever encounter. In making these predictions, it can be helpful to imagine actual physical inscriptions existing alongside a larger system of abstract objects ('strings') which witness

[28]Perhaps one might argue that what explains usefulness is really just syntactic consistency, but I would respond by suggesting logical coherence provides a better explanation for the practical usefulness of formal system than mere consistency, and perhaps even that logical coherence provides the best explanation for the syntactic consistency of some formal systems.

all logically possible ways putting together finitely many letter inscriptions chosen from the relevant finite alphabet.

First note that even if all the inscriptions we encounter are relatively short, the most efficient way for us to recognize patterns in what inscriptions are physically possible can plausibly involve recognizing the logical possibility of strings of arbitrary finite size. Many 'closure principles' which smoothly (help) predict the facts about what short strings are physically possible will have the consequence that very long strings are logically possible - even strings which are too long to physically realize given the number of fundamental particles in the universe. Take, for example, the principle that for any logically possible inscription it is logically possible for there to be a 'doubled' inscription which concatenates that inscription with itself. Given the truth of principles like this, a scenario in which there are objects witnessing all logically possible choices of how to concatenate letters will be a scenario in which there are infinitely many different abstract objects. Our methods of reasoning about logical possibility for infinite collections can be tested and corrected by the consequences they have for what this infinite collection of all strings would have to be like (and thereby, indirectly, for what physical string inscriptions are possible)[29].

Next I claim that hypothesising a space of a countably infinitely many strings which contains witnesses to all these logical possibility facts is practically useful. While one could understand all these claims about strings in

[29]Admittedly the presumption that logico-mathematical reality is (in some sense) uniform plays a key role here. I think the very same expectation that reality is simple and uniform which drives access worries, tells us to expect that logical possibility facts are uniform. We can, of course, learn that large collections fail to be like small ones in certain specific ways (c.f. Hilbert's hotel) by using our methods which effectively do presume that large collections are like small ones. But I think that the discovery of such caveats and exceptions shouldn't inspire general skepticism about our scientific-induction presumption that there are elegant general laws which apply everywhere in the logical case – any more than discovering an unexpected new type of star should lead us to reject the project of astronomy (trying to learn laws that constrain the universe as a whole based on some finite portion of it we have access to) as impossible or unreliable.

terms of modal statements (e.g. 'it's logically necessary that if s is a string it's possible to have a string that concatenates s with s') it's more useful to think in terms of a single logically possible structure containing all finite strings (which is itself an infinite collection). This provides provides another kind of pressure to recognize the logical possibility of an infinite structure.

5.3. **On the Limits of Our Knowledge of Logical Possibility.** A final worry concerns our access to facts about logical possibility involving larger infinite collections. Perhaps one can explain our accuracy in reasoning about countably infinite collections as above. Yet capturing intuitively correct truth conditions for statements of set theory (via the structuralist consensus) requires evaluating claims about the logical possibility of scenarios involving uncountably many objects. Thus, one might worry that principles of reasoning which are shaped to elegantly predict and explain what is logically possible for finite and countably infinite collections cannot account for the degree of logical (and hence mathematical) knowledge which we actually have.

Admittedly the presumption that logico-mathematical reality is (in some sense) uniform plays a key role in the picture being advocated here. And admittedly my proposal (that our logical possibility knowledge ultimately arises from something like generalization of laws discovered by dealing with simple concretely realizable cases) suggest certain limits to this knowledge. Perhaps it suggests that this knowledge should become rather sparser as we consider larger structures.

However, as [2] argues, this prediction actually fits rather well with what we observe in actual mathematics. A critic might suggest that taking elegant generalization from 'small' (finite and countable) collections to general principles which yield truths about the much larger collections considered in pure mathematics is like saying that inference to the best explanation plus observations of birds in New Mexico explains our possession of true beliefs about birds in Canada as well. One should expect the knowledge attainable in this way to be rather limited.

I want to respond by accepting the analogy, and the conclusion that (if my story is true) our knowledge of higher mathematics should be rather limited – but saying that this prediction fits the apparent state of human mathematical knowledge fairly well, so it is a benefit rather than a flaw of my account. Even in the case of birds, we can arrive at some true beliefs about birds by generalization from one environment to another. If we discovered tomorrow that some new island which had never yet been visited by explorers contained birds, I think we would reasonably expect many facts to carry over: any birds on that island would breathe oxygen, that they would have hollow bones etc. Our expectations about birds on this island would just be more sparse and less confident than our beliefs about birds in locations that we have observed.

But, this is just what happens with regard to our beliefs about logical possibility and large collections: as one moves from logical possibility facts concerning finite collections to those concerning countably infinite collections (like the natural numbers), and then uncountable collections (like the sets) our beliefs do get more sparse and less confident. For example, the continuum hypothesis [30] (CH) is a fairly simple statement involving sets of (relatively) small infinite size, yet it is known that (assuming ZFC is consistent) both the truth and the falsity of CH are compatible with ZFC. Our

[30] The continuum hypothesis states that there are no sets whose cardinality is intermediate between the cardinality of the real numbers and that of the natural numbers. See [] pg 176-186 for the proof that the continuum hypothesis is independent of the Zermelo-Fraenkel axioms.

beliefs about what large infinite collections of objects and relations are logically possible are also frequently less confident than our beliefs about what finite and countable collections of objects are logically possible[31]. Sociologically, mathematicians are frequently much more confident in their claims about numbers, sets of numbers and sets of sets of numbers than in the distinctive claims of set theory about what much larger patterns of mathematical objects would have to be like.

Thus, I think this last worry points to something that is an attractive feature rather than a flaw of the account at hand: it explains why we have relatively sparse beliefs about what's logically possible with respect to large collections, and hence relatively sparse beliefs about the corresponding facts concerning higher set theory.

However, this naturally raises the question of how much accuracy about higher set theory mathematics can be explained in the way I propose. Personally, I'm inclined to think that logical possibility facts are sufficiently uniform for the process of reflective equilibrium outlined above to account for our accurate recognition of the logical possibility of ZFC (and thus all the theorems of mainstream set theory). But someone who presumes less uniformity within logical possibility facts (and takes a more skeptical attitude to the higher reaches of set theory) may accept accept my solution to access worries concerning the mathematics they believe in (e.g. number theory), while denying that kind of uniformity we can expect from logical possibility is strong enough to get us to accurate principles about ZFC.

6. Conclusion

In this paper I have fleshed out an answer to access worries which combines popular structuralist ideas about the connection between mathematics and

[31]Think of countable choice as opposed to disputes over large cardinal axioms.

logic with an appeal to dealings with concretea to explain how we could get suitably powerful general principles for reasoning about logical possibility to explain accuracy about mathematics.

If we think about our conceptions of mathematical structures in terms of something like conditional logically possibility (as I have argued above that it is natural to do), then inductive generalization from dealings with concreta to explain our possession of good general methods of reasoning about logical possibility which let us systematically adopt only coherent mathematical posits.

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