Supplementary material for *A Logical Foundation for Potentialist Set Theory*

E Auxiliary Definitions

We adopt the convention that $x \leq y$ abbreviates $x < y \vee x = y$.

Definition E.1. *$<$ linearly orders the objects which satisfy W*

- *$<$ is antisymmetric*

  $(\forall x)(\forall y)\neg(x < y \wedge y < x)$
- *$<$ is transitive*

  $(\forall x)(\forall y)(\forall z)(x < y \wedge y < z \rightarrow x < z)$
- *$<$ is total on W*

  $(\forall x)(\forall y)(W(x) \wedge W(y) \rightarrow x < y \vee y < x)$

Definition E.2 (Well-Order). *A two-place relation $<$ **well-orders** the objects which satisfy W (equivalently $(W, <)$ is a **well-order**) iff*

- *$<$ linearly orders the objects which satisfy W*
- *Least Element Condition:* $\Box_{W,<}[(\exists x \mid W(x))P(x) \rightarrow (\exists y \mid W(y) \wedge P(y))(\forall z \mid W(z) \wedge P(z))(y \leq z))]$

We will also say that $\leq$ well-orders W just if the above conditions are satisfied when $x < y$ is replaced with $x < y \wedge x \neq y$.

Note that the Least Element Condition guarantees that if P is a predicate on W (i.e., $P(x) \rightarrow W(x)$) then if P is non-empty it has a $<$ least element.

The following lemma tells us that if V is an initial segment, the relations in V' agree with V whenever both are defined, every set in V' is available at an ordinal in V' and the ordinals in V' are downward closed in V.

Lemma E.1 (Initial Segment Lemma). *Suppose V is an initial segment and the relations* $(\text{set}', \text{ord}', \in', <', @')$ *satisfy the following*

- $(\forall o)(\text{ord}'(o) \rightarrow \text{ord}(o))$
- $(\forall u, o \mid \text{ord}'(u) \wedge \text{ord}'(o))(u <' o \leftrightarrow u < o)$
- $(\forall u, o \mid \text{ord}'(o))(u < o \rightarrow \text{ord}'(u))$
- $(\forall x)(\text{set}'(x) \rightarrow \text{set}(x))$

- $(\forall x, y \mid \text{set}'(x) \land \text{set}'(y))(x \in' y \leftrightarrow x \in y)$
- $(\forall x \mid \text{set}'(x))(\forall u \mid \text{ord}'(u))(@'(x,u) \leftrightarrow @(x,u))$
- $(\forall x)(\text{set}'(x) \leftrightarrow (\exists u)[\text{ord}'(u) \land @'(x,u)])$

then $V' \leq V$.

Proof. This fact can be shown fairly straightforwardly by checking through definitions. ■

Definition E.3. *A well-order* $(W', <')$ *extends a* $(W, <)$*, written* $(W', <') \geq (W, <)$ *if*

- $(\forall o)(W(o) \to W'(o))$
- $(\forall o, u \mid W(o) \land W(u))(o < u \leftrightarrow o <' u)$
- $(\forall o \mid W(o))(\forall u <' o \mid W'(u))(W(u))$

Note that if $(W', <')$ is a well order and $(W', <')$ extends $(W, <)$ then $(W, <)$ is also a well order.

F Set Theoretic Mimicry

I will now describe how to use the familiar formal background of set theory to *mimic* intended truth conditions for statements in a language containing the logical possibility operator ◊ alongside usual first order logical vocabulary (where distinct relation symbols R_1 and R_2 always express distinct relations) as follows.

> A formula ψ is true relative to a model $\mathcal{M}$ ($\mathcal{M} \vDash \psi$) and an assignment ρ which takes the free variables in ψ to elements in the domain of $\mathcal{M}$[1] just if:

- $\psi = R_n^k(x_1 \ldots x_k)$ and $\mathcal{M} \vDash R_n^k(\rho(x_1), \ldots, \rho(x_k))$.
- $\psi = x = y$ and $\rho(x) = \rho(y)$.
- $\psi = \neg\phi$ and ϕ is not true relative to $\mathcal{M}, \rho$.
- $\psi = \phi \land \psi$ and both ϕ and ψ are true relative to $\mathcal{M}, \rho$.
- $\psi = \phi \lor \psi$ and either ϕ or ψ are true relative to $\mathcal{M}, \rho$.
- $\psi = \exists x \phi(x)$ and there is an assignment ρ' which extends ρ by assigning a value to an additional variable v not in ϕ and $\phi[x/v]$ is true relative to $\mathcal{M}, \rho'$[2].

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[1] Specifically: a partial function ρ from the collection of variables in the language of logical possibility to objects in $\mathcal{M}$, such that the domain of ρ is finite and includes (at least) all free variables in ψ

[2] As usual (?) $\phi[x/v]$ substitutes v for x everywhere where x occurs free in ϕ

- $\psi = \Diamond_{R_1 \dots R_n} \phi$ and there is another model $\mathcal{M}'$ which assigns the same tuples to the extensions of $R_1 \dots R_n$ as $\mathcal{M}$ and $\mathcal{M}' \vDash \phi$.[3]

Note that this means that $\bot$ is not true relative to any model $\mathcal{M}$ and assignment ρ.

If we ignore the possibility of sentences which demand something coherent but fail to have set models because their truth would require the existence of too many objects, we could then characterize logical possibility as follows:

> **Set Theoretic Approximation**: A sentence in the language of logical possibility is true (on some interpretation of the quantifier and atomic relation symbols of the language of logical possibility) iff it is true relative to a set theoretic model whose domain and extensions for atomic relations captures what objects there are and how these atomic relations actually apply (according to this interpretation) and the empty assignment function ρ.

G Natural Deduction System

Definition G.1 (Natural Deduction For Logical Possibility). *A proof in this system consists of inferences in accordance with the following rules. Note that we will abuse notation and call*

(Ass)
Φ Ass. [n] can be written down on a line n (in any context). This corresponds to making a new assumption.

(FOL)
ϕ FOL $(1_1, \dots l_n)$ $[a_1, \dots, a_j]$ can be written on line i iff all of the following hold

- $i > l_1, \dots l_n$,
- $i, l_1, \dots l_n$ *are all in the exactly the same* $\Diamond$ *contexts*
- *Φ is derivable from sentences on lines $1_1, \dots l_n$ via FOL*
- *$[a_1, \dots, a_j]$ is the union of all the line numbers cited as assumptions on lines $l_1, \dots l_n$. Note that if a_i is starred on line l_j then it appears starred here.*

(→I)
$\Psi \to \Phi(m) \to I[a_1, \dots a_k]$ can be written down on a line $n > m$ in a context C iff line m contains the sentience Φ with assumptions $[a_1, \dots a_k, a_{k+1}]$ and Ψ is the formula on line a_{k+1}. This inference rule corresponds to discharging the assumption (made on line a_{k+1} that Ψ

[3] As usual, I am taking □ to abbreviate ¬?¬

Logical Axioms
Φ X can be written down on any line, provided X is the name of a logical axiom schema, and Φ is an instance of this axiom schema.

Logical Axioms 2(shortcut)
Ψ X $[a_1, \dots, a_n, b_1^*, \dots, b_m^*]$ can be written down on line $i > l$ provided X is the name of some logical axiom schema which $\Phi \rightarrow \Psi$ instantiates and Φ is the sentence on line l with $[a_1, \dots, a_j]$ indicated as assumptions.

Inn ◊ I
One can indent and write down $^{\diamond}$ | $\Phi\{L\}$ In $\diamond I[n^*]$ *on any line n iff we have $\lozenge_{\mathcal{L}}\, \Phi$ on some line $m < n$ in some context C, such that any further $\diamond$ contexts entered between lines m and n have already been exited. (Doing this amounts to beginning a new Inner $\diamond$ and I will say that one is thereby 'opening' and entering a new inner $\diamond$ context C', for which C is the immediate parent).*

Importing
One can write down $\Phi(m)$ Importing *n in any line $n > m$ in an Inner ◊ Context C, provided that Φ is content restricted to the relations $\{\mathcal{L}\}$ held fixed by context C, and m is a line in the immediate parent context to C with the sentence Φ..*[4]

Inner ◊ E
: $\lozenge_{\mathcal{L}}\, \Psi$ $m, k - l$ Inn $\diamond E$ *$[a_1, \dots, a_n, b_1^*, \dots, b_m^*]$ can be written down on a line $n > m, k, l$ in the present context, provided that*

- *line m is also in the same context and contains a sentence of the form $\lozenge_{\mathcal{L}}\, \Phi$,*
- *The] lines $k - l$ belong to an inner diamond context opened on line k from the present context by citing line m. Furthermore, line l must not belong to any further nested ◊ context besides the one opened on line k, i.e., only one $\diamond$ context can be exited via this rule.*
- *line l asserts the sentence Ψ under assumptions $[k_1^*, \dots k_j^*]$. The fact all the k_i^* are starred ensures that all $[k_1^*, \dots k_j^*]$ are introduced by Importing or the Diamond introduction rule.*
- *$i \in \{a_1, \dots a_n\}$ ($i^* \in \{b_1^*, \dots b_m^*\}$) just if there is some k_j^* which cites line l as a justification, i.e., l occurs in parenthesis on line k_j^* and i (i^*) is cited as an assumption on line l. Note that i (i^* will always be a line in the current context since the Importing and Inner ◊ I rules must cite lines in the parent context.*

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[4] *Note this isn't quite the importing rule, but (as we will see below) you can think of it as justified by the fact that you could have conjoined the phi you want to import with the psi in the diamond statement which our current diamond context reasons about.*

G.1 Correctness For Natural Deduction

To see that the above natural deduction system only permits the creation of proofs allowed by the official formal system proposed in Chapter *8*, we offer the following inductive argument.

Let Φ_i be the sentences appearing on line i of the proof with $[l_1^i, \dots, l_{n_i}^i, k_1^{i*}, \dots, k_{m_i}^{i*}]$ appearing on line i as well. Let Γ_i be the total collection of assumptions under which Φ_i is asserted, i.e. $\Gamma_i = \{\Phi_j \mid (\exists p)\left[j = l_{n_p} \vee j = k_{m_p}\right]\}$. We claim that for all lines i in the proof $\Gamma_i \vdash \Phi_i$. It is evident this suffices to prove the desired claim.

By induction, assume the claim holds for all lines $i' < i$. It is evident the claim also holds for line i unless i is an instance of In◊E. So suppose i is an instance of In ◇E. We now show that Γ_i gathers up all the assumptions of lines citing importing or initiating the inner ◇ context and use the *Proposition B.1 (The Inner Diamond Lemma)* to prove the inductive claim.

By the rules for Inn◊E, we are working inside a Diamond context introduced via Inn ◇I on some earlier line i_1. Let $\Theta = \Phi_{i_1}$. By the rules for Inn ◇I, there is some prior line i_0 in the parent context to line i_1 with $\Phi_{i_0} = \lozenge_{\mathcal{L}}\,\Theta$ cited as a justification for Inn ◇I. Let $\Gamma = \Gamma_{i_0}$, i.e., the set of Φ_j such that line j is listed in brackets (starred or unstared) on line i_0. By inductive hypothesis we have $\Gamma \vdash \lozenge_{\mathcal{L}}\,\Theta$.

By the rules for Inn◊E Φ_i has the form the form $\lozenge_{\mathcal{L}}\,\Psi$ and cites some earlier line i_Ψ as justification. Furthermore, the rules for Inn ◇E guarantees that the numbers in brackets on line i_Ψ are all starred.

Let $\Gamma' = \Gamma_{i_\Psi} - \{\Theta\}$. By the inductive hypothesis $\Gamma', \Theta \vdash \Phi$. If j is one of the line numbers in brackets on line i_Ψ it is starred and thus either $j = i_1$ (and $\Phi_{i_1} = \Theta$) or line j was introduced into the same context as line i_Ψ by the importing rule. Let $\hat{\jmath}$ be the line cited as a justification for line j when $j \neq i_1$. By the rules governing importing $\hat{\jmath}$ occurs in the same context as line i (the parent context of line j), $\Phi_{\hat{\jmath}} = \Phi_j$ and Φ_j is content restricted to $\mathcal{L}$. Hence every sentence in Γ' is content restricted to $\mathcal{L}$.

Thus, by the *Proposition B.1 (The Inner Diamond Lemma)* it follows that $\Gamma, \Gamma' \vdash \lozenge_{\mathcal{L}}\,\Theta$.

By the rule for Inn◊E, we have $\Gamma_i \supseteq \Gamma = \Gamma_{i_0}$. If $\Upsilon \in \Gamma'$ then $\Upsilon = \Phi_j$ for some j listed in brackets on line i_Ψ. By the rule for Inn ◇E we have $\Gamma_i \supseteq \Gamma_{\hat{\jmath}}$ where $\Phi_j = \Phi_{\hat{\jmath}}$. By the inductive hypothesis, $\Gamma_{\hat{\jmath}} \vdash \Phi_{\hat{\jmath}} = \Upsilon$. Hence, $\Gamma_i \vdash \Gamma \cup \Gamma'$ and by transitivity it follows that $\Gamma_i \vdash \lozenge_{\mathcal{L}}\,\Theta = \Phi_i$ as desired.

H Useful Corollaries to Axioms

H.1 More Basic Box Lemmas

Here are more □ versions of basic ◊ axioms and lemmas above.

Lemma H.1 (□ Ignoring). *If θ is content-restricted to a list of relations $\mathcal{L}, \mathcal{R}$ which doesn't include any relations in list $\mathcal{S}$, then $\Box_{\mathcal{L},\mathcal{S}}\,\theta \to \Box_{\mathcal{L}}\,\theta$.*

Proof. Assume the antecedent, i.e., $\neg \Diamond_{\mathcal{L},\mathcal{S}} \neg \theta$. $\Diamond$ Ignoring (*Axiom 8.3*) tells us that $\Diamond_{\mathcal{L}} \neg\theta \rightarrow \Diamond_{\mathcal{L},\mathcal{S}} \neg\theta$. So we can infer $\neg \Diamond_{\mathcal{L}} \neg\theta$. ■

1	$\Box_{\mathcal{L},\mathcal{S}}\theta$	[1]
2	$\neg\Diamond_{\mathcal{L},\mathcal{S}}\neg\theta$	[1]
3	$\Diamond_{\mathcal{L}}\neg\theta \rightarrow \Diamond_{\mathcal{L},\mathcal{S}}\neg\theta$	$\Diamond$ Ignoring
4	$\neg\Diamond_{\mathcal{L}}\neg\theta$	2,3 FOL [1]
5	$\Box_{\mathcal{L}}\theta$	[1]
6	$\Box_{\mathcal{L},\mathcal{S}}\theta \rightarrow \Box_{\mathcal{L}}\theta$	1-5 $\rightarrow$I

Lemma H.2 (Box Closure). *If* $\Phi \vdash \Psi$ *then* $\Box_{\mathcal{L}} \Phi \rightarrow \Box_{\mathcal{L}} \Psi$

Proof. Note that by *Lemma B.3 (Box Elimination)*

$$\Box_{\mathcal{L}} \Phi \vdash \Psi$$

The conclusion now follows by *Lemma B.2 (Box Introduction)* ■

Lemma H.3 (Box Importing). *If* Φ *is content restricted to* $\mathcal{L}$ *then* $\Phi \wedge \Box_{\mathcal{L}} \Psi \rightarrow \Box_{\mathcal{L}} \Phi \wedge \Psi$

Proof. Infer Ψ from $\Box_{\mathcal{L}} \Psi$ via *Lemma B.3 (Box Elimination)* thus

$$\Phi \wedge \Box_{\mathcal{L}} \Psi \vdash \Phi \wedge \Psi$$

The conclusion now follows by *Lemma B.2 (Box Introduction)* (as Φ is content restricted to $\mathcal{L}$). ■

Lemma H.4 (Expanded $\Box$ Elimination). *Suppose* $R_1, \dots, R_n$ *are distinct relations not in* $\mathcal{L}$ *and* $R'_1, \dots, R'$ *are (potentially non-distinct) relations (potentially in* $\mathcal{L}$*) of the same arity as* $R_1, \dots, R_n$ *then*

$$\Box_{\mathcal{L}} \Theta \rightarrow \Theta[R_1/R'_1 \dots R_n/R'_n]$$

Proof. Provided $R'_1, \dots, R'_n$ don't appear in Θ, this claim can be derived simply by using Relabeling (*Axiom 8.5*) to get $\neg \Diamond_{\mathcal{L}} \neg\Theta \leftrightarrow \neg \Diamond_{\mathcal{L}} \neg\Theta[R_1/R'_1 \dots R_n/R'_n]$ and hence $\Box_{\mathcal{L}} \Theta\, [R_1/R'_1 \dots R_n/R'_n]$ then applying *Lemma B.3 (Box Elimination)* . We can also use Relabeling to go from $\Box_{\mathcal{L}} \Theta$ to $\Box_{\mathcal{L}} \Theta\, [R_{1*}/R'_{1*} \dots R_{n*}/R'_{n*}]$ making all of the replacements

R_i/R'_i for R'_i that don't occur in Θ. Thus, to prove the claim, it is enough to prove it on the assumption that all of $R'_1, \ldots, R'_n$ do appear in Θ[5].

Suppose this claim fails. In this case we have $\neg\Theta[R_1/R'_1 \ldots R_n/R'_n]$. Let $R''_1, \ldots, R''_n$ be unused relations of the same arity as $R'_1, \ldots, R'_n$. Now repeatedly enter Inner Diamond contexts and apply Simple Comprehension (*Axiom 8.4*) to infer

$$\Diamond_{\mathcal{L}} \ldots \Diamond_{\mathcal{L}} \, [\neg\Theta[R_1/R'_1 \ldots R_n/R'_n] \wedge \\ (\forall \overrightarrow{z_1})(R''_1(\overrightarrow{z_1}) \leftrightarrow R'_1(\overrightarrow{z_1})) \wedge \ldots \wedge (\forall \overrightarrow{z_n})(R''_n(\overrightarrow{z_n}) \leftrightarrow R'_n(\overrightarrow{z_n}))]$$

Entering all these contexts (i.e., applying *Proposition B.1 (Inner Diamond)* n-times) and applying *Axiom 8.6 (Importing)* in each, we may infer that $\Box_{\mathcal{L}} \Theta$ applies in this context. Now, as $R''_1, \ldots, R''_n$ don't appear in Θ, we may apply the version of the lemma already verified above and then use the equivalence of these relations to the relations $R'_1, \ldots, R'_n$ to derive a contradiction, which we may then export to complete the proof. Note that we omit the formal proof that if R'_i and R''_i hold of the same tuples then $\Theta[R_1/R'_1 \ldots R_n/R'_n] \leftrightarrow \Theta[R_1/R''_1 \ldots R_n/R''_n]$. Note that in cases where Θ includes nested possibility claims, this argument requires either a complex inductive argument or use of *Theorem I.1 (Isomorphism Lemma)* which is proved (without use of this result) in below (we use Simple Comprehension (*Axiom 8.4*) again to introduce an identity relation $Z(x, y)$ and argue that $R'_1, \ldots R'_n \cong_Z R''_1, \ldots R''_n$ and then use the *Theorem I.1 (Isomorphism Lemma)* to infer $\Theta[R_1/R'_1 \ldots R_n/R'_n] \leftrightarrow \Theta[R_1/R''_1 \ldots R_n/R''_n]$). ■

H.2 Box and Diamond Simplification Lemmas

Lemma H.5 (Box Simplification). $\Box_{\mathcal{L}_0}(\psi \rightarrow \Box_{\mathcal{L}_1}(\phi \rightarrow \theta)) \rightarrow \Box_{\mathcal{L}_0}(\psi \wedge \phi \rightarrow \theta)$

Proof. Assume $\Box_{\mathcal{L}_0}(\psi \rightarrow \Box_{\mathcal{L}_1}(\phi \rightarrow \theta))$. We now prove $\psi \wedge \phi \rightarrow \theta$ follows from this assumption.

Assume $\psi \wedge \phi$. By *Lemma 4.4 (Box Elimination)* we may infer $\psi \rightarrow \Box_{\mathcal{L}_1}(\phi \rightarrow \theta)$ and thus $\Box_{\mathcal{L}_1}(\phi \rightarrow \theta)$. By another application of *Box Elimination* and modus ponens, we infer θ. Hence, we have

$$\Box_{\mathcal{L}_0}(\psi \rightarrow \Box_{\mathcal{L}_1}(\phi \rightarrow \theta)) \vdash \psi \wedge \phi \rightarrow \theta$$

But since the sentence on the left is content restricted to $\mathcal{L}_0$ applying *Lemma 4.3 (Box Introduction)* gives us

$$\Box_{\mathcal{L}_0}(\psi \rightarrow \Box_{\mathcal{L}_1}(\phi \rightarrow \theta)) \vdash \Box_{\mathcal{L}_0}(\psi \wedge \phi \rightarrow \theta)$$

which trivially entails the desired conclusion. ■

Or, alternately, we can present the proof as follows.

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[5] Note that no relation can both appear in Θ and not appear in Θ, allowing us to safely split the substitution into two pieces even if some of the substituted relations are repeated.

1	$\Box_{\mathcal{L}_0}(\psi \to \Box_{\mathcal{L}_1}(\phi \to \theta))$	[1]
2	$\Box$ $[\mathcal{L}_0]$	
3	$\phi_1 \wedge \phi_2$	[3]
4	$\Box_{\mathcal{L}_0}(\psi \to \Box_{\mathcal{L}_1}(\phi \to \theta))$	1, import [1]
5	$\psi \to \Box_{\mathcal{L}_1}(\phi \to \theta)$	4 $\Box$ E [1]
6	$\Box_{\mathcal{L}_1}(\phi \to \theta)$	3,5 FOL [1,3]
7	$\phi \to \theta$	6 $\Box$ E [1,3]
8	θ	3,7 FOL [1,3]
9	$\phi_1 \wedge \phi_2 \to \theta$	3,8 $\to$I [1]
10	$\Box_{\mathcal{L}}(\phi_1 \wedge \phi_2 \to \theta)$	2-5 $\Box$ I [1]

Lemma H.6 (Diamond Simplification). *If Ψ is content restricted to $\mathcal{L}_\Psi \supset \mathcal{L}$, Φ is content restricted to $\mathcal{L}_\Phi$ and $\mathcal{L}_\Psi \cap \mathcal{L}_\Phi \subset \mathcal{L}'$ then*

$$\Diamond_{\mathcal{L}}(\Psi \wedge \Diamond_{\mathcal{L}'}\Phi) \to \Diamond_{\mathcal{L}}(\Psi \wedge \Phi)$$

Or, equivalently, $\Box_{\mathcal{L}_0}(\psi \wedge \phi \to \theta) \to \Box_{\mathcal{L}_0}(\psi \to \Box_{\mathcal{L}_1}(\phi \to \theta))$

Proof. Note that the equivalently statement is merely the contraposative of the above claim (using the definition of $\Box$ as $\neg \Diamond \neg$) taking Φ to be $\phi \wedge \neg\theta$ and Ψ to be ψ. Thus it's enough to prove the main claim.

Consider an arbitrary $\Phi, \Psi, \mathcal{L}, \mathcal{L}', \mathcal{L}_\Phi, \mathcal{L}_\Phi$ satisfying the assumptions above.

$$\Diamond_{\mathcal{L}}(\Psi \wedge \Diamond_{\mathcal{L}'}\Phi)$$

Enter this $\Diamond_{\mathcal{L}}$ context. We know that Φ and $\Diamond_{\mathcal{L}'}\Phi$. As Φ is content restricted to $\mathcal{L}_\Phi$, by $\Diamond$ Ignoring (*Axiom 8.3*) we can add to the subscript of $\Diamond_{\mathcal{L}'}\Phi$ any relations that don't occur in $\mathcal{L}_\Phi$. By our assumption that $\mathcal{L}_\Psi \cap \mathcal{L}_\Phi \subset \mathcal{L}'$, all relations in $\mathcal{L}_\Psi$ but not already in $\mathcal{L}'$ don't occur in $\mathcal{L}_\Phi$. So we can add these relations to the subscript to get $\Diamond_{\mathcal{L}_\Psi \cup \mathcal{L}'}\Phi$. And as Ψ is content restricted to $\mathcal{L}_\Psi$, we can use Importing (*Axiom 8.6*) to infer that $\Diamond_{\mathcal{L}_1 \cup \mathcal{L}_\Psi}(\Psi \wedge \Phi)$. Hence by $\Diamond$ Ignoring (*Axiom 8.3*) and $\mathcal{L}_\Psi \supset \mathcal{L}$, we have $\Diamond_{\mathcal{L}}(\Psi \wedge \Phi)$.

So leaving this $\Diamond_{\mathcal{L}}$ context, (completing our Inner Diamond *Proposition B.1* argument) we have

$$\Diamond_{\mathcal{L}}(\Diamond_{\mathcal{L}}(\Psi \wedge \Phi))$$

Finally, we can apply *Axiom 8.2 (Diamond Elimination)* to conclude $\Diamond_{\mathcal{L}}(\Psi \wedge \Phi)$ as the latter sentence is content restricted to $\mathcal{L}$).

1	$\Diamond_{\mathcal{L}}\,(\Psi \wedge \Diamond_{\mathcal{L}'}\Phi)$	[1]
2	$\Diamond$ $\Psi \wedge \Diamond_{\mathcal{L}'}\Phi$	[$\mathcal{L}$] Inner $\Diamond$ [1]
3	Ψ	2 FOL [1]
4	$\Diamond_{\mathcal{L}'}\Phi$	2 FOL [1]
5	$\Diamond_{\mathcal{L}' \cup \mathcal{L}_\Psi}\Phi$	3 Ignoring [1]
6	$\Diamond_{\mathcal{L}' \cup \mathcal{L}_\Psi}\Phi \wedge \Psi$	3,5 Importing [1]
7	$\Diamond_{\mathcal{L}}\Phi \wedge \Psi$	9 Reducing[1]
8	$\Diamond_{\mathcal{L}}\Diamond_{\mathcal{L}}(\Phi \wedge \Psi)$	1, 2-10 Inn $\Diamond$ E [1]
9	$\Diamond_{\mathcal{L}}(\Phi \wedge \Psi)$	11 $\Diamond$ E [1]

■

H.3 Multiple Definitions Lemmas

Often we will want to apply several instances of Simple Comprehension (*Axiom 8.4*), *Proposition 8.1 (Simplified Choice)*, *Axiom 8.9 (Modal Comprehension)* or *Axiom 8.12 (Choice)* in sequence to specify the application of a series of relations $R_1 ... R_n$. The multiple definitions lemma enables us to do this at once, without entering a new $\Diamond$ context for each deployment of one of the above principles.

Lemma H.7 (Multiple Definitions Lemma). *Suppose that Ψ holds and that, for each i with $0 \leq i \leq n$, Φ_i is such that $\Diamond_{\mathcal{L},R_0,\ldots,R_{i-1}}\Phi_i$ is the conclusion got by applying* Axiom 8.4 (Simple Comprehension), Proposition 8.1 (Simplified Choice), Axiom 8.9(Modal Comprehension) *or* Axiom 8.12 (Choice) *to specify the possible application of some a relation R_i (so R_i doesn't appear in Ψ or in any Φ_j with $j < i$) and $\Psi \wedge \Phi_0 \wedge \ldots \Phi_{i-1}$ entails the antecedent of the respective lemmas. Then $\Diamond_{\mathcal{L}}(\Psi \wedge \Phi_0 \wedge \ldots \Phi_n)$.*

Proof. A trivial induction (letting Ψ in Simple Comprehension (*Axiom 8.4*), *Proposition 8.1 (Simplified Choice)*, *Axiom 8.9 (Modal Comprehension)* or *Axiom 8.12 (Choice)* be $\Psi \wedge \Phi_0 \wedge \ldots \wedge \Phi_{i-1}$) lets us conclude that

$$\Diamond_{\mathcal{L}}\ \Diamond_{\mathcal{L},R_0} \ldots \Diamond_{\mathcal{L},R_0,\ldots,R_{n-1}}(\Psi \wedge \Phi_0 \wedge \ldots \Phi_n)$$

Applying *Lemma B.8 (Diamond Collapsing)* n times yields the desired result. ■

To see how this lemma applies, suppose P is a non-empty two place relation and we wish to consider the possibility ($\Diamond_P$) that some predicate Q selects a single x such that $(\exists y)P(x,y)$, e.g., for a proof by contradiction. In this example *Lemma H.7 (Multiple Definitions)* lets us pack together successive applications of: Simple Comprehension (*Axiom 8.4*) to define $O(x)$ to hold iff $(\exists y)P(x,y)$ and then *Proposition 8.1 (Simple Choice)* to define Q to apply to a unique element of O.

$$(\exists x)(\forall y)[P(y,x) \rightarrow (\forall z)(P(y,z) \rightarrow z = x)]$$

We could now simply say: As $(\exists x)(\exists y)P(x,y)$, by the *Lemma H.7 (Multiple Definitions)* together with *Axiom 8.4 (Simple Comprehension)* and *Proposition 8.1 (Simplified Choice)* we can ($\Diamond_P$) have $(\exists x)(\exists y)P(x,y)$, while O applies to all x such that $(\exists y)P(x,y)$ and Q applies to a unique x such that $(\exists y)P(x,y)$. Indeed, when O isn't itself relevant to further argument, we will sometimes omit mention of it and just say that: we can ($\Diamond_P$) have $(\exists x)(\exists y)P(x,y)$, remain true while Q applies to a single object x such that $(\exists y)P(x,y)$.

H.4 Singleton Lemma

By Possible Powerset (*Axiom 8.11*), it's possible to supplement the objects satisfying Ext($\mathcal{L}$) with a disjoint collection of objects coding all possible classes of elements from Ext($\mathcal{L}$).

The following lemma verifies the simple fact that every object x satisfying Ext($\mathcal{L}$) has a unique singleton. First, however, we adopt the following notation.

Definition H.1 (Singleton). *Let $\{x\}_C$ denote the element satisfying C containing only x. In particular, let $y = \{x\}_C$ abbreviate the formula $x \underset{C}{\in} y \wedge (\forall z)(z \underset{C}{\in} y \rightarrow z = x)$*

Lemma H.8 (Singleton Lemma). *If $\mathcal{C}(C, \in_C$,Ext$(\mathcal{L}))$ then $(\forall x \mid \text{Ext}(\mathcal{L})(x))(\exists! y \mid y = \{x\}_C)$*

Or, equivalently, the map from x to $\{x\}$ is functional (note this doesn't imply that the map must be given by some relation).

Proof. The uniqueness claim is immediate from the extensionality clause in the definition of $\mathcal{C}(C, \in_C$,Ext$(\mathcal{L}))$. We thus need only prove the existence claim, i.e.

$$(\forall x \mid \text{Ext}(\mathcal{L})(x))(\exists y \mid y = \{x\}_C)$$

Let $\mathcal{L}'$ be $\mathcal{L} \cup \{C, \in_C, F\}$.

Suppose the claim fails for some x. By the Multiple Definitions Lemma (*Lemma H.7*) (packing together an application of Simple Comprehension (*Axiom 8.4*) and *Proposition 8.1 (Simplified Choice)*, it is possible ($\Diamond_{\mathcal{L}'}$) for $\mathcal{C}(C, \in_C, \mathcal{L})$ to remain true while Q applies to a unique x witnessing this failure.

Enter this ($\Diamond_{\mathcal{L}'}$) context. By the fatness condition in the definition of $\mathcal{C}$ (with respect to the predicate Q), we can derive the existence of $\{x\}_Q$ giving us this contradiction. Exporting the contradiction via *Axiom 8.2 (Diamond Elimination)* yields the desired result. ■

/ Isomorphism Theorem

In this section we will prove a generalization of the following, very intuitive, principle. If $\langle R_1, \ldots R_m \rangle \cong \langle {R_1}', \ldots R_m; \rangle$ and Φ is a sentence about $\langle R_1, \ldots R_m \rangle$ then it holds iff $\Phi[R_1/R'_1, \ldots, R_m/R'_m]$ holds. We formalize this as follows.

Theorem I.1 (Isomorphism Theorem). *Suppose that*

- $\langle R_1, \ldots, R_m \rangle \cong_f \langle {R_1}', \ldots, {R_m}' \rangle$
- *ϕ is content restricted to $R_1, \ldots, R_m$*
- *Each[6] R'_i is either identical to R_i or is distinct from all R_i and doesn't appear in ϕ.*
- *f doesn't appear in ϕ and f isn't identical to any R_i or R'_i*
- $\phi' = \phi[R_1/R'_1, \ldots, R_m/R'_m]$

then $(\forall a_1, \ldots, a_n \mid \text{Ext}(R_1, \ldots, R_m)(a_1) \wedge \ldots \text{Ext}(R_1, \ldots, R_m)(a_n))$
$\phi(a_1, \ldots, a_n) \leftrightarrow \phi'(f(a_1), \ldots, f(a_k))$

We will prove this lemma inductively[7]. The main difficulty in doing this will be showing the truth of the claim for $\Diamond_L \psi$ statements, given it holds for all formulas with fewer logical possibility operators than $\Diamond_L \psi$. If we could assume that ψ was content restricted to some list of relations, the proof would be relatively straightforward. But the fact that $\Diamond_L \psi$ is content restricted doesn't guarantee that ψ is. For $\Diamond_L \phi$ can be content restricted to some list of relations $R_1 \ldots R_n$ (and hence satisfy the assumptions of the lemma) in cases where the sentence ϕ is not content restricted to any list of relations, so our inductive hypothesis tells us nothing about it directly.

Accordingly, I will first prove the following lemma, which shows that we can associate every non-content restricted sentence Φ with a content restricted version of this sentence $\widehat{\Phi}$ such that $\vdash \Diamond_{\mathcal{L}} \Phi \leftrightarrow \Diamond_{\mathcal{L}} \widehat{\Phi}$. Our strategy here will be to define $\widehat{\Phi}$ to be the result of restricting all quantifiers appearing at the top level of Φ to some new predicate U (plus an additional technical assumption). We then argue that if $\Diamond_{\mathcal{L}} \Phi$ (or $\Diamond_{\mathcal{L}} \widehat{\Phi}$) it is also possible that Φ ($\widehat{\Phi}$) obtains and everything satisfies U, allowing us to infer the possibility of $\widehat{\Phi}$ (Φ).

Lemma I.1 (Content Restricted Equivalent Lemma). *Let Φ be a sentence in the language of logical possibility, $\mathcal{L}$ a list of relations, U a predicate not occurring in Φ or $\mathcal{L}$, and $\mathcal{L}_\Phi$ the set of relations appearing at the top level in Φ (i.e., without being enclosed in any other $\Diamond$ operator, where relations subscripted by top level $\Diamond$ operators don't count as enclosed). Then there is a sentence $\widehat{\Phi}_U$ such that*

1. *$\widehat{\Phi}_U$ is (explicitly) content restricted to $\mathcal{L}_\Phi \cup \{U\} \cup \mathcal{L}$*

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[6] *Furthermore, we implicitly assume that the R_i are distinct relations as well as the R'_i.*

[7] I presented an informal version of this proof in [Berry 2015].

2. *(Equivalence)* $\Diamond_{\mathcal{L}} \widehat{\Phi}_U \leftrightarrow \Diamond_{\mathcal{L}} \Phi$ *is a theorem (in the system used in this book).*

3. *(Same Depth)* $\widehat{\Phi}_U$ *contains the same number of* $\Diamond$ *operators as* Φ *does.*

4. *If* $R'_1, \dots R'_l, U'$ *are distinct relations of the same arity as* $R_1, \dots, R_l, U$ *not occurring in* Φ *nor equal to any* R_i *or to* U *then* $\widehat{\Phi}_U[R_1/R'_1, \dots R_l/R'_l, U/U'] = (\Phi[R_1/\widehat{R'_1, \dots} R_l/R'_l])_{U'}$

Proof. Let

$$\widehat{\Phi}_U = \Phi^U \wedge (\forall x \mid \mathrm{Ext}(\mathcal{L}_\Phi, \mathcal{L})(x))(U(x)) \wedge (\exists x)U(x)$$

where Φ^U is formed from Φ by taking all quantifiers appearing outside of any $\Box$ or $\Diamond$ operator and restricting them to U, i.e., $(\forall x)\theta$ becomes $(\forall x)(U(x) \to \theta(x))$ and $(\exists x)\theta$ becomes $(\exists x)(U(x) \wedge \theta)$. We leave the formal inductive statement of this operation to the reader.

Note the obvious fact (provable by an induction on formula complexity which we omit) that if every object satisfies U then restricting quantifiers to U makes no difference to the truth of a claim, i.e., $(\forall x)U(x) \to (\Phi^U \leftrightarrow \Phi)$.

It is evident by the definition of Content Restriction *(Definition 7.2)* that $\widehat{\Phi}_U$ is content restricted to $\mathcal{L}_\Phi \cup \{U\} \cup \mathcal{L}$, giving us clause 1 above. Both clauses 3 and 4 are obvious from the construction of $\widehat{\Phi}_U$. So we must only demonstrate that $\Diamond_{\mathcal{L}} \widehat{\Phi}_U \leftrightarrow \Diamond_{\mathcal{L}} \Phi$

$(\to)$ Suppose $\Diamond_{\mathcal{L}} \widehat{\Phi}_U$. Enter this $\Diamond_{\mathcal{L}}$ context. As

$$(\forall x \mid \mathrm{Ext}(\mathcal{L}_\Phi, \mathcal{L})(x))(U(x)) \wedge (\exists x)U(x)$$

we can apply *Axiom 8.8 (Cutback)* to infer $\Diamond_{\mathcal{L}_\Phi, \mathcal{L}, U} (\forall x)(U(x))$. Enter this $\Diamond_{\mathcal{L}_\Phi, \mathcal{L}, U}$ context. We can import $\widehat{\Phi}_U$, by the fact, noted above, that it is content restricted to $\mathcal{L}_\Phi, U$. This gives us Φ^U and as $(\forall)U(x) \to (\Phi^U \leftrightarrow \Phi)$ is a theorem we can infer Φ. Leaving both $\Diamond$ contexts lets us infer $\Diamond_{\mathcal{L}} \Diamond_{\mathcal{L}_\Phi, \mathcal{L}, U} \Phi$ which, by Diamond Collapsing (*Lemma B.4*), gives us $\Diamond_{\mathcal{L}} \Phi$.

$(\leftarrow)$. Suppose $\Diamond_{\mathcal{L}} \Phi$. Enter this $\Diamond_{\mathcal{L}}$ context. As U doesn't occur in Φ or $\mathcal{L}$, by Simple Comprehension (*Axiom 8.4*) we can derive

$$\Diamond_{\mathcal{L}} \Phi \wedge (\forall x)(U(x) \leftrightarrow x = x)$$

Entering this $\Diamond_{\mathcal{L}}$ context we can infer $(\forall x \mid \mathrm{Ext}(\mathcal{L}_\Phi \cup \mathcal{L})(x))(U(x))$ and as $\vdash (\forall)U(x) \to (\Phi^U \leftrightarrow \Phi)$ we may infer Φ^U. Lastly, by first order logic we can derive $(\exists x)(x = x)$ and hence $(\exists x)(U(x))$, giving us all the conjuncts of $\widehat{\Phi}_U$. Leaving both $\Diamond$ contexts/closing both inner $\Diamond$ arguments gives us $\Diamond_{\mathcal{L}} \Diamond_{\mathcal{L}} \widehat{\Phi}_U$. So applying Diamond Collapsing (*Lemma B.8*) gives us $\Diamond_{\mathcal{L}} \widehat{\Phi}_U$ as desired. ∎

We will also need one more lemma, which lets us specify logically possible extensions for new relations $P_1', \dots, P_n'$, so that an isomorphism between some original structures $\langle R_1, \dots R_l \rangle$ and $\langle R'_1, \dots R'_l \rangle$ can be extended to an isomorphism between the larger structures $\langle R_1, \dots R_l, P_1, \dots, P_n \rangle$ and $\langle R'_1, \dots R'_l, P'_1, \dots, P'_n \rangle$.

Lemma I.2 (Isomorphism Extension Lemma). *Suppose that $\langle R_1, \dots R_m\rangle \cong_f \langle R'_1, \dots R'_m\rangle$ all R_i, P_i, R'_i are in $\mathcal{L}$, and g and the $P'_1, \dots P'_n$ are 'fresh' relations distinct from all $\mathcal{L}, f$ and each other. Then*

$$\Diamond_{\mathcal{L},f}\left[\langle R_1, \dots R_m, P_1, \dots, P_n\rangle \underset{g}{\cong} \langle R'_1, \dots R'_m, P'_1, \dots, P'_n\rangle \wedge \big(\forall x \mid Ext(R_1 \dots R_m)(x)\big)\big(f(x) = g(x)\big)\right.$$

Proof. Suppose that the conditions of the lemma are satisfied. By the Possible Powerset axiom (*Axiom 8.11*) we have

$$\Diamond_{\mathcal{L},f}\, \mathcal{C}\,(C, \underset{C}{\in}, \text{Ext}(R_1, \dots, R_m, R'_1, \dots, R'_m, P_1, \dots P_n))] \qquad (I1)$$

Enter this $\Diamond_{\mathcal{L},f}$ context.

Recall that $y = \{x\}_C$ abbreviates $x \in_C y \wedge (\forall z)(z \in_c y \to z = x)$. And by the *Lemma H.8* $\{x\}_C$ is functional, 1-1 and defined on all of $\text{Ext}(R_1, \dots, R_m, R'_1, \dots, R'_m, P_1, \dots P_n)$.

By *Axiom 8.4 (Simple Comprehension)* and the *Lemma H.7 (Multiple Definitions)* we can infer that it is logically possible($\Diamond_{\mathcal{L},f,C,\in_C}$) to have the interior of *(I1)*, which we know to hold true in our current context, remain true while defining both g and P'_i for $1 \le i \le m$ as follows

$$g(x) = \begin{cases} f(x) & \text{if Ext}(R_1, \dots R_m)(x) \\ \{x\}_C & \text{otherwise} \end{cases}$$
$$P'_i(g(x_1), \dots, g(x_{k_i})) \leftrightarrow P_i(x_1, \dots x_{k_i})$$

Enter this $\Diamond_{\mathcal{L},f,C,\in_C}$ context. We know g is $1-1$ on $\text{Ext}(R_1, \dots, R_m, P_1, \dots P_n)$ by the fact that f and the singleton relation are $1-1$ and there are no x, y with $\{x\}_C = f(y)$ as the classes introduced by $\mathcal{C}(C, \in_C, F)$ are disjoint from all the objects satisfying $\text{Ext}(R_1, \dots, R_m, R'_1, \dots, R'_m, P_1, \dots P_n)$.

We know g is functional and defined on all of $\text{Ext}(R_1, \dots, R_m, P_1, \dots P_n)$ by the fact that f is functional and defined on all of $\text{Ext}(R_1, \dots, R_m)$, and the 'singleton relation' is functional and defined on all of $\text{Ext}(P_1, \dots P_n)$.

All other facts needed for this to be an isomorphism follow immediately from the imported fact that f is an isomorphism and the P'_i are defined to satisfy the definition of isomorphism. ∎

We now turn to the proof of the theorem.

Theorem I.1. (Isomorphism Theorem) *Suppose that*

- $\langle R_1, \dots, R_m\rangle \cong_f \langle R_1', \dots, R_m'\rangle$
- *ϕ is content restricted to $R_1, \dots, R_m$*
- *Each[8] R'_i is either identical to R_i or is distinct from all R_i and doesn't appear in ϕ.*

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[8] *Furthermore, we implicitly assume that the R_i are distinct relations as well as the R'_i.*

- *f doesn't appear in ϕ and f isn't identical to any R_i or R'_i*
- $\phi' = \phi[R_1/R'_1, \dots, R_m/R'_m]$

then $(\forall a_1, \dots, a_n \mid \text{Ext}(R_1, \dots, R_m)(a_1) \land \dots \text{Ext}(R_1, \dots, R_m)(a_n))$
$$\phi(a_1, \dots, a_n) \leftrightarrow \phi'(f(a_1), \dots, f(a_k))$$

Proof. We first observe that it is enough to prove the implies direction of the claim since as the reverse direction follows by[9] application of the forward direction to f^{-1}.

We now prove the forward direction by induction on the structure of ϕ. We assume that the claim is true (for all m and relations $R_1, \dots, R_m$) both for all subformulas of ϕ and for all formula that contain strictly fewer logical possibility operators than ϕ. We now attempt to verify the claim for ϕ.

The base case, where ϕ is an atomic formula is straightforward, as are the cases where ϕ is a truth-functional combination of other formula.

Now assume that $\phi(a_1, \dots a_k)$ begins with an existential quantification. As $\phi(a_1, \dots a_k)$ is content restricted to $R_1, \dots, R_m$, we may assume that if ϕ is of the form[10] $(\exists x \mid \text{Ext}(R_1 \dots R_m)(x))(\psi(a_1, \dots a_k, x))$. If ϕ holds then for some b satisfying $\text{Ext}(R_1 \dots R_m)(b)$ we have $\psi(a_1, \dots a_k, b)$. Thus, by the inductive assumption we have $\psi'(f(a_1), \dots, f(a_k), f(b))$. And as, by the definition isomorphism (*Definition 7.4*), f bijects $\text{Ext}(R_1, \dots, R_m)$ with $\text{Ext}(R'_1, \dots, R'_m)$, we have

$$\begin{aligned} \phi'(f(a_1), \dots f(a_k)) &\leftrightarrow (\exists x \mid \text{Ext}(R_1, \dots, R_m)(x))(\psi'(f(a_1), \dots f(a_k), f(x))) \\ &\leftrightarrow (\exists x \mid \text{Ext}(R_1', \dots, R_m')(x))(\psi'(f(a_1), \dots f(a_k), x)) \end{aligned}$$

. The case where ϕ is a universal formula is already handled, as we identify $\forall$ with $\neg\exists\neg$.

Finally, consider the case where $\phi = \Diamond_{\mathcal{L}_0} \Phi$. Suppose that ϕ is true (note that Φ must be a sentence so we need not worry about free variables) and $\langle R_1, \dots, R_m \rangle \cong_f \langle R'_1, \dots, R'_m \rangle$. We need to prove that $\Diamond_{\mathcal{L}'_0} \Phi'$, where $\mathcal{L}'_0$ is the result of replacing each R_i in $\mathcal{L}_0$ with R'_i. Note that as ϕ is content restricted to $R_1, \dots, R_m$ we must have $\mathcal{L}_0 \subset \{R_1, \dots, R_m\}$ and, (by renumbering if necessary) we can assume that $\mathcal{L}_0 = \{R_1, \dots, R_l\}$ and $\mathcal{L}'_0 = \{R'_1, \dots, R'_l\}$. Note that we have $\langle R_1, \dots, R_l \rangle \cong_f \langle R'_1, \dots, R'_l \rangle$.

Our first step will be use the *Lemma I.1 (Content Restricted Equivalent)* to infer that $\Diamond_{\mathcal{L}_0} \widehat{\Phi}_U$ where $\widehat{\Phi}_U$ is a version of Φ which is content restricted to $\mathcal{L}_0, \mathcal{L}_\Phi, U$ where $\mathcal{L}_\Phi$ is the set of

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[9] Specifically, we can argue that it's possible to have a relation $g = f^{-1}$ while maintaining all the other assumptions of the theorem. We may then apply the proof to g to infer the possibility of the theorem's conclusion and export it to infer the reverse direction.

[10] Note that it is enough to prove the claim for explicitly content restricted formulas and the result for implicitly content restricted formulas is immediate.

relations appearing at the top level of Φ and U is a predicate distinct from all relations hitherto mentioned.

In particular, by *clause 2* of *Lemma I.1* we can infer the following from ϕ.

$$\Diamond_{\mathcal{L}_0} \widehat{\Phi}_U$$

We wish to import the fact that $\langle R_1, \dots, R_l \rangle \cong_f \langle R'_1, \dots, R'_l \rangle$ and to that end we expand the set of relations held fixed to $\mathcal{L}_0, \mathcal{L}_0', f$. By assumption, no relation in $\mathcal{L}_0' - \mathcal{L}_0$ appears in ϕ, nor does f. Hence,

$$\big((\{f\} \cup \mathcal{L}_0') - \mathcal{L}_0\big) \cap (\mathcal{L}_0 \cup \mathcal{L}_\Phi \cup \{U\}) = \emptyset$$

Thus, as $\widehat{\Phi}_U$ is content restricted to $\mathcal{L}_0, \mathcal{L}_\Phi, \{U\}$, by $\Diamond$ Ignoring (*Axiom 8.3*) we can infer

$$\Diamond_{\mathcal{L}_0, \mathcal{L}_0', f} \widehat{\Phi}_U$$

Enter the $\Diamond_{\mathcal{L}_0, \mathcal{L}_0', f}$ context provided by senence above and import the fact that $\langle R_1, \dots, R_l \rangle \cong_f \langle R'_1, \dots, R'_l \rangle$. To apply the inductive hypothesis we need to construct a g extending f that isomorphicly maps $\mathcal{L}_\Phi, U$ to some $\mathcal{L}_\Phi', U'$. Letting $P_1, \dots, P_n, U$ be the relations (if any) in $\mathcal{L}_\Phi, U$ not in $\mathcal{L}_0$ and $P'_1, \dots, P'_n, U'$ some previously unmentioned relations of the same arity as $P_1, \dots, P_n, U$ we invoke the *Lemma I.2 (Possible Isomorphism Extending)* with some new relation g to infer

$$\Diamond_{\mathcal{L}_0, U, f, \mathcal{L}_0', \mathcal{L}_\Phi} \left(\langle R_1, \dots R_l, P_1, \dots, P_n, U \rangle \underset{g}{\cong} \langle R'_1, \dots R'_l, P'_1, \dots, P'_n, U' \rangle \right)$$

Enter this additional $\Diamond_{\mathcal{L}_0, U, f, \mathcal{L}'_0, \mathcal{L}_\Phi}$ context and import $\widehat{\Phi}_U$ (it is content restricted to $\mathcal{L}_\Phi, U$). We are finally in a position to apply the inductive hypothesis. For by *clause 3* in *Lemma I.1 (Content Restricted Equivalent)*, $\widehat{\Phi}_U$ has the same number of $\Diamond$ operators as Φ and thus strictly fewer than ϕ does. Moreover, as U', g, P'_i were all chosen to be distinct relations thet don't appear in Φ. And, by assumption any R'_i which does appear in Φ is identical to the corresponding R_i. So, by inductive hypothesis, we can infer

$$\Diamond_{\mathcal{L}_0, U, f, \mathcal{L}'_0, \mathcal{L}_\Phi} \widehat{\Phi}_U \, [R_1/R'_1, \dots R_l/R'_l, U/U', P_1/P'_1, \dots, P_n/P'_n]$$

By applying Reducing (*Lemma B.4*) we infer

$$\Diamond_{\mathcal{L}_0, U, f, \mathcal{L}'_0} \widehat{\Phi}_U \, [R_1/R'_1, \dots R_l/R'_l, U/U', P_1/P'_1, \dots, P_n/P'_n]$$

As no P'_i appears in $\mathcal{L}_0, \mathcal{L}'_0, U, f$ we can use Relabeling (*Axiom 8.5*) to substitute P_i in for P_i' yielding

$$\Diamond_{\mathcal{L}_0, U, f, \mathcal{L}'_0} \widehat{\Phi}_U \, [R_1/R'_1, \dots R_l/R'_l, U/U', P_1/P'_1, \dots, P_n/P'_n][P'_1/P_1, \dots, P'_n/P_n]$$

Which simplifies to

$$\Diamond_{\mathcal{L}_0,U,f,\mathcal{L}'_0} \widehat{\Phi}_U\, [R_1/R'_1, \dots R_l/R'_l, U/U']$$

Renumbering if necessary, we may assume that $R_1', \dots, R_k'$ don't appear in $\mathcal{L}_0$ and that $R_{k+1} = R'_{k+1} \dots R_l = R'_l$. As R_i/R_i a null operation this gives us

$$\Diamond_{\mathcal{L}_0,U,f,\mathcal{L}'_0} \widehat{\Phi}_U\, [R_1/R'_1, \dots R_k/R'_k, U/U']$$

Since $R'_i, 1 \leq i \leq k$ doesn't occur in ϕ by *clause 4 (replacing)* of *Lemma I.1(Content Restricted Equivalent Theorem)* we may know that

$$\widehat{\Phi}_U[R_1/R'_1, \dots R_l/R'_l, U/U'] = (\Phi[R_1/\widehat{R'_1, \dots} R_l/R'_l])_{U'}$$

Dropping out of the enclosing $\Diamond$ contexts gives us

$$\Diamond_{\mathcal{L}_0,\mathcal{L}_0',f}\ \Diamond_{\mathcal{L}_0,U,f,\mathcal{L}'_0} \widehat{\Phi}_U\, [R_1/R'_1, \dots R_k/R'_k, U/U']$$

A combination of Reducing (*Lemma B.4*) and Diamond Collapsing (*Lemma B.8*) yields.

$$\Diamond_{\mathcal{L}'_0} (\Phi[R_1/\widehat{R'_1, \dots} R_k/R'_k])_{U'}$$

And by clause 2 (equivalence) in *Lemma I.1 (Isomorphism Lemma)* this implies (reinstating the null substitutions of $R_{k+1}/R'_{k+1} \dots R_l/R'_l$)

$$\Diamond_{\mathcal{L}'_0} \Phi\, [R_1/R'_1, \dots R_l/R'_l]$$

Since no R'_i with $i > l$ appear in $\mathcal{L}'_0$ or $\Phi[R_1/R'_1, \dots R_l/R'_l]$ and no R_i appears in $\mathcal{L}'_0$ (unless $R'_i = R_i$ in which case the substitution below is the null operation) we make invoke Relabeling (*Axiom 8.5*) on $R'_i, i > l$ to derive

$$\Diamond_{\mathcal{L}'_0} \Phi\, [R_1/R'_1, \dots R_l/R'_l, \dots R_m/R'_m]$$

But this is just our desired conclusion that $\Diamond_{\mathcal{L}'_0} \Phi\,'$ completing our proof. ∎

J. $\boldsymbol{PA}_{\Diamond}$ and Infinite Well Ordering Lemmas

In this appendix I will show that *Axiom 8.10 (Infinity)* (together with my other inference rules) implies the Infinite Well-Ordering Theorem (*Theorem J.1*) below. While *Axiom 8.10 (Infinity)* was chosen to be as simple as possible and only asserts the possibility of a scenario with a successor function to justify the set theoretic axiom of infinity we must derive the possibility of an infinite well-order.

As the particular infinite well-ordering whose possibility we establish will be ω, this Lemma will let us quickly prove that $\Diamond\ (PA_{\Diamond})$. That is, it's logically possible for there to be some objects which (when considered under some relations) satisfy the categorical description of the natural numbers structure discussed in Section *J.3*.

Recall that the Infinity axiom says the following.

Axiom J.1 (Infinity). $\Diamond\, \Psi$ *where Ψ is the conjunction of the following claims:*

1. *The successor of an object is unique* $(\forall x)(\forall y)(\forall y')[S(x,y) \wedge S(x,y') \rightarrow y = y']$

2. *successor is one-to-one* $(\forall x)(\forall y)(\forall x')(S(x,y) \wedge S(x',y) \rightarrow x = x')$

3. *there is a unique object that has a successor and isn't the successor of anything* $(\exists! x\colon (\exists y)S(x,y) \wedge (\forall y)\neg S(y,x))$

4. *everything that is a successor has a successor* $(\forall x)[(\exists y)S(y,x) \rightarrow (\exists z)S(x,z)]$

5. S *is anti-reflexive:* $(\forall x)(\forall y)[S(x,y) \rightarrow \neg S(y,x)]$

So, (speaking informally) *Axiom 8.10 (Infinity)* says that we could have an infinite collection objects related by a successor relation S in a successor-like way (remember we often abbreviate $S(x,y)$ by $S(x) = y$). We will now derive two useful consequences from this claim.

- That there could be an infinite well ordering $W, \leq$.

- That $\mathbb{N}, S$ could apply to objects satisfying the (conditional possibility version of) the Peano Axioms

Theorem J.1 (Infinite Well-Ordering Theorem). *It is logically possible for there to be a non-empty well ordering with no maximal element. And we can further require that every element of this well ordering has a maximal predecessor. That is, the conjunction of the following claims is logically possible.*

1. *(Well-ordered)* $\leq$ *well-orders*[11] *the objects satisfying* W

2. *(Non-Empty)* $(\exists x)(W(x))$

3. *(No Maximal Element)* $(\forall x \mid W(x))(\exists y \mid W(y))(x < y)$

4. *(Least Element) There is a unique minimal element, i.e.,* $(\exists! z)(\forall x)(x \not< z \wedge W(z))$.

5. *(Discreteness)* $(\forall b)(b = 0 \vee (\exists a < b)(\forall z < b)(z \leq a)$ *Every non-zero element has a maximal predecessor.*

Given a relation S satisfying the conditions in *Axiom 8.10 (Infinity)*, our approach will be to define W to apply to the smallest class closed under S containing the 0 element. Note that in this proof we use 0 to abbreviate the unique element that has a successor but isn't a successor. We will demonstrate that this corresponds to the unique minimal element referenced in part 4 *(least element)* of the theorem. We then define $x \leq y$ to hold if every successor closed class containing x contains y.

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[11] *Remember we defined the notion of well-order both for* $<$ *and* $\leq$ *relations.*

J.1 Constructing the well-ordering

The *Axiom 8.10 (infinity)* tells us that $\lozenge\, \Omega$, where Ω abbreviates the following sentence (which is content restricted to S).

$$\begin{aligned}
&(\forall x)(\forall y)(\forall y')[S(x,y) \wedge S(x,y') \rightarrow y = y'] \wedge \\
&(\forall x)(\forall y)(\forall x')(S(x,y) \wedge S(x',y) \rightarrow x = x') \wedge \\
&(\exists! x \mid (\exists y)S(x,y) \wedge (\forall y)\neg S(y,x)) \wedge \\
&(\forall x)[(\exists y)S(y,x) \rightarrow (\exists z)S(x,z)] \wedge \\
&(\forall x)(\forall y)[S(x,y) \rightarrow \neg S(y,x)]
\end{aligned}$$

Note that the first conjunct implies that S if functional. Enter this $\lozenge$ context. Using the Possible Powerset Axiom (*Axiom 8.11*) it's possible ($\lozenge_S$) to have a layer of classes over the objects related by S.

Enter this $\lozenge_S$ context. Ω must remain true, as it is content restricted to S, so we have:

$$\Omega \wedge \mathcal{C}\left(C, \underset{C}{\in}, \text{Ext}(S)\right). \qquad (J1)$$

Next, by using *Lemma H.7 (Multiple Definitions)* to pack together successive applications of *Axiom 8.4 (Simple Comprehension)* it is possible ($\lozenge_{S,C,\in_C}$) that equation (J1) remains true along with the conjunction of the following four facts.

$$\begin{aligned}
&D(x) \leftrightarrow \text{Ext}(S)(x) \qquad (J2) \\
&(\forall x)[SC(x) \leftrightarrow C(x) \wedge (\forall z)(\forall z')(z \in x \wedge S(z,z') \rightarrow z' \in x)] \qquad (J3) \\
&(\forall x)[W(x) \leftrightarrow D(x) \wedge (\forall k \mid C(k))[0 \in k \wedge SC(k) \rightarrow x \in k])]. \qquad (J4) \\
&(\forall x)(\forall y)(x \leq y \leftrightarrow D(x) \wedge D(y) \wedge (\forall k)[x \in k \wedge SC(k) \rightarrow y \in k]) \qquad (J5)
\end{aligned}$$

Informally, the above equations have the following effects.

- (J2) ensures D serves as a shorthand for Ext(S)
- (J3) ensures $SC(d)$ holds just if d is successor closed.
- (J4) ensures The relation $x \leq y$ holds for elements in D just if y is in every successor closed class that x is in.
- (J5) ensures $W(x)$ holds just if x is an element of every successor closed class containing 0 (the unique element that has a successor but isn't one).

So leaving all $\lozenge$ contexts and letting Δ denote the conjunction of the above four equations yields

$$\lozenge \lozenge_S \lozenge_{S,C,\underset{C}{\in}} \left[\Omega \wedge \mathcal{C}(C, \underset{C}{\in}, \text{Ext}(S)) \wedge \Delta\right]$$

And by Diamond Collapsing (*Lemma B.4*) this implies

$$\lozenge \left[\Omega \wedge \mathcal{C}\left(C, \underset{C}{\in}, \text{Ext}(S) \right) \wedge \Delta \right] \qquad (J6)$$

This completes the construction of $W, \leq$, our non-empty well ordering with no maximal element. We must now check that this logically possible scenario really behaves as advertised.

J.2 Verification

We now enter the $\lozenge$ from equation *(J6)* giving us the following equation.

$$\Psi: \Omega \wedge \mathcal{C}\left(C, \underset{C}{\in}, \text{Ext}(S) \right) \wedge \Delta. \qquad (J7)$$

Before verifying $W, \leq$ has the required features I will first prove a utility lemma showing that for appropriate formulas γ we can always find a class whose members are exactly those picked out by γ.

Lemma J.1 (Class Comprehension). *Suppose that* $\mathcal{C}(C, \in_C, \text{Ext}(\mathcal{L}))$ *and* $\gamma(x)$ *is a* $\lozenge$ *and* $\square$ *free formula content restricted to some* $\mathcal{L}' \supset \mathcal{L} \cup \{C, \in_C\}$ *with only* x *free rendering. Then* $(\exists g \mid C(g))(\forall x)(x \underset{C}{\in} g \leftrightarrow \text{Ext}(\mathcal{L})(x) \wedge \gamma(x))$

Proof. Suppose the assumptions of the lemma hold. By Simple Comprehension (*Axiom 8.4*) it's possible ($\lozenge_{\mathcal{L}'}$), while keeping the assumptions of the lemma true, that the following holds

$$(\forall x)(G(x) \leftrightarrow \gamma(x))$$

Enter this $\lozenge_{\mathcal{L}'}$ context we can unpack $\mathcal{C}(C, \in_C, \text{Ext}(\mathcal{L}))$ giving us

$$\square_{C, \underset{C}{\in}, \text{Ext}(\mathcal{L})} (\exists g)[C(g) \wedge (\forall x)((D(x) \wedge G(x)) \leftrightarrow x \in g)]$$

By *Lemma B.3 (Box Elimination)* we can deduce that

$$(\exists g)[C(g) \wedge (\forall x)((D(x) \wedge G(x)) \leftrightarrow x \in g)]$$

so by the fact that $(\forall x)(G(x) \leftrightarrow \gamma(x))$ we can deduce

$$(\exists g)[C(g) \wedge (\forall x)(x \in g \leftrightarrow D(x) \wedge \gamma(x))]$$

By assumption this sentence is implicitly content restricted to $\mathcal{L}'$ we can exit the current $\lozenge_{\mathcal{L}'}$ context and invoke *Axiom 8.2 (Diamond Elimination)* to conclude

$$(\exists g)[C(g) \wedge (\forall x)(x \in g \leftrightarrow \text{Ext}(\mathcal{L})(x) \wedge \gamma(x))]$$

holds in our original scenario. ■

We now enter the $\lozenge$ context from *(J6)* and assuming $\Omega \wedge \mathcal{C}(C, \in_C, \text{Ext}(S)) \wedge \Delta$ prove a series of lemmas that, together, will satisfy the elements of the Infinite Well-Ordering Theorem (

Theorem J.1). Note that in this situation since $D(x) \leftrightarrow \mathrm{Ext}(S)(x)$ we can invoke the above lemma using $D(x)$ in place of $\mathrm{Ext}(S)(x)$.

Lemma J.2 (Non-Emptiness). $(\exists x)W(x)$

Proof. By equation *(J4)* $W(x)$ holds iff $D(x) \wedge (\forall k \mid C(k))[0 \in_C k \wedge SC(k) \rightarrow x \in_C k])$. As $\mathrm{Ext}(S)(0)$ by equation *(J1)* we have $D(0)$. And clearly $0 \in_C k \rightarrow 0 \in_C k$. Hence $W(0)$. This verifies that $(W, \leq)$ satisfies *clause 2 (non-empty)* of *Theorem J.1 (Isomorphism Theorem)*. ∎

Lemma J.3 (Reflexivity). $(\forall x)(W(x) \rightarrow x \leq x)$

Proof. By *(J4)* If $W(x)$ then $D(x)$. And by *(J5)* $x \leq y$ iff $D(x) \wedge D(y) \wedge (\forall k)[x \in k \wedge SC(k) \rightarrow y \in k])$ so $x \leq x$. ∎

Lemma J.4 (Transitivity). $(\forall x, y, z \mid W(x) \wedge W(y) \wedge W(z))(x \leq y \wedge y \leq z \rightarrow x \leq z)$

Proof. Consider arbitrary x, y and z satisfying W such that $x \leq y \wedge y \leq z$. Suppose that $x \in k \wedge SC(k)$. Then as $x \leq y$ by *(J5)* we have $y \in k \wedge SC(k)$ and as $y \leq z$ we can infer $z \in k$. Hence $(\forall k)(x \in k \wedge SC(k) \rightarrow z \in k)$. Thus, by *(J5)*, $x \leq z$. ∎

Lemma J.5 (Totality). $(\forall x)(\forall y)[W(x) \wedge W(y) \rightarrow x \leq y \vee y \leq x]$

Proof. First, we introduce some abbreviations

$$\mathrm{CMP}(x, y) \leftrightarrow x \leq y \vee y \leq x$$
$$\mathrm{ALLCMP}(x) \leftrightarrow (\forall y \mid W(y))\mathrm{CMP}(x, y)$$

By *Lemma J.1 (Class Comprehension)* above applied to the formula

$$\gamma(x) \leftrightarrow D(x) \wedge \mathrm{ALLCMP}(x)$$

we can infer that there is a unique object g such that $C(g)$ and

$$(\forall x \mid W(x))[x \in g \leftrightarrow D(x) \wedge \mathrm{ALLCMP}(x)]$$

It is thus enough to show that $W(x) \rightarrow x \in g$. Now clearly $0 \in g$ since, by equation *(J4)* every x satisfying $W(x)$ is in every successor closed class containing 0 which is the requirement equation *(J5)* gives for $0 \leq x$. Therefore, by equation *(J4)*, if g is successor closed then g contains every element satisfying W. Suppose g is not successor closed. That is

$$(\exists x)\left(\mathrm{ALLCMP}(x) \wedge \neg\mathrm{ALLCMP}\big(S(x)\big)\right) \qquad (J8)$$

Let x witness the truth of the above equation and let y witness the failure of $\mathrm{ALLCMP}(S(x))$, i.e., y satisfies $\neg\mathrm{CMP}(S(x), y)$. More specifically, note that by using the *Lemma H.7 (Multiple Definitions)* to pack together applications of Simple Comprehension (*Axiom 8.4*) and *Proposition 8.1 (Simplified Choice)*, we can ($\lozenge_{W,SC,C,\in_C,D,\leq}$) have the predicate Q_x apply to a unique x witnessing the truth of the existential claim in *(J8)* and Q_y applying to a unique y which $S(x)$ is not comparable to.

Enter this $\lozenge_{W,SC,C,\in_C,D,\leq}$ context.

Now, by assumption ALLCMP(x) hence $x \leq y \vee y \leq x$. Suppose $y \leq x$. If k is a successor closed class containing y then by equation *(J5)* we have $x \in_C k$ and as k is successor closed $S(x) \in_C k$. Hence, by equation *(J5)* $y \leq S(x)$ contradicting the fact that $\neg$CMP$(S(x), y)$.

Suppose $x \leq y$. As $\neg$CMP$(S(x), y)$ we can't have $S(x) \leq y$ so there must be some successor closed class k containing $S(x)$ but not y. Now, by the same reasoning as above (using Q_x, Q_y to avoid quantifying into the $\Diamond$), possibly ($\Diamond_{W,SC,C,\in_C,D,\leq,Q_x,Q_y}$) Q_k applies to a single class k witnessing this fact. Enter this $\Diamond_{W,SC,C,\in_C,D,\leq,Q_x,Q_y}$ context. Now, applying *Lemma J.1 (Class Comprehension)* (as Q_k and Q_x apply only to objects satisfying D) there is some class $k' = k \cup \{x\}$, i.e., a k' such that

$$(\forall z)(z \underset{C}{\in} k' \leftrightarrow (\exists k)(z \underset{C}{\in} Q_k(k) \vee Q_x(z))$$

Or, equivalently,

$$(\forall z)(z \underset{C}{\in} k' \leftrightarrow z \underset{C}{\in} k \vee z = x)$$

As k was successor closed and contained $S(x)$, it is trivial to see that k' is successor closed. By our choice of k we have $y \notin_C k$, so either $y \notin_C k'$ or $y = x$. However, since $y \nleq x$ $y = x$ is ruled out by the fact that $\leq$ is reflexive (*Lemma J.3 Reflexivity*). So $y \notin_C k'$. But then k' is a successor closed class containing x but not y which, by equation *(J5)*, contradicts the assumption that $x \leq y$. Exporting the contradiction, we can thus conclude that g is successor closed completing the proof of comparability. ■

Lemma J.6 (Maximal Predecessor). $(\forall a)(\forall z \leq S(a))(z = S(a) \vee z \leq a)$

Note that as 0 is the unique element without a successor, this suffices prove the claim in *clause 5 (Discreteness)* of the theorem that every non-zero element has a maximal predecessor since if $b \neq 0$ then, for some a, $S(a) = b$. We also know $a \leq b$ as every successor closed class containing a contains $S(a)$ hence this lemma entails

$$(\forall b)(b = 0 \vee (\exists a < b)(\forall z < b)(z \leq a)$$

Proof. Suppose that the lemma fails. Then, by the same reasoning as above using the the Multiple Definitions Lemma (*Lemma H.7*) to pack together applications of *Proposition 8.1*, and Simple Comprehension (*Axiom 8.4*) we can ($\Diamond_{W,SC,C,\in_C,D,\leq}$) have Q_a applying to a single object a and Q_z applying to a single object z such that $z < S(a)$ but not $z \leq a$. We enter this $\Diamond_{W,SC,C,\in_C,D,\leq}$ context and import any of the assumptions we need.

Since $\neg(z \leq a)$, by equation *(J5)* there must be some successor-closed class containing z but not a. By the Multiple Definitions Lemma (*Lemma H.7*) it is possible ($\Diamond_{W,SC,C,\in_C,D,\leq,Q_a,Q_z}$) that Q_k picks out a single class k witnessing this fact. Enter this $\Diamond_{W,SC,C,\in_C,D,\leq,Q_a,Q_z}$ context.

Now using Q_a, Q_z, Q_k, apply *Lemma J.1 (Class Comprehension)* to derive the existence of a k' such that $x \in_C k' \leftrightarrow x \in_C k \wedge x \neq S(a)$ (i.e., $k' = k - \{S(a)\}$). As k is successor closed and k doesn't contain a, and $S(a)$ isn't the successor of any other object, it follows that k' is successor closed. By our choice of k we have $z \in_C k$. So, as $z \neq a$ (by *Lemma J.3 (Reflexivity)* and our choice of z) we can infer $z \in_C k'$. Thus k' is a successor closed class which contains z but not $S(a)$. However, by *(J5)*, this contradicts the fact that $z \leq S(a)$ and exporting the contradiction establishes the result. ■

Lemma J.7 (Well-Ordering). $\Box_{W,\leq}[\quad (\exists x)(K(x) \wedge W(x)) \rightarrow (\exists x')(K(x') \wedge W(x') \wedge (\forall y)[K(y) \wedge W(y) \rightarrow x' \leq y])]$

Proof. Suppose not. Then we have

$$\Diamond_{W,\leq}[\quad (\exists x)(K(x) \wedge W(x)) \wedge \neg(\exists x')(K(x') \wedge W(x') \wedge (\forall y)[K(y) \wedge W(y) \rightarrow x' \leq y])]$$

By $\Diamond$ Ignoring (*Axiom 8.3*) we can deduce the corresponding $\Diamond_{SC,C,S,\in_C,W,\leq,D}$ claim. Entering this $\Diamond_{SC,C,S,\in_C,W,\leq,D}$ context we can import Ψ and deduce $(\exists x)(K(x) \wedge W(x))$ and

$$(\forall x')(K(x') \wedge W(x') \rightarrow (\exists y)[K(y) \wedge W(y) \wedge x' \nleq y]) \qquad (J9)$$

Now by the *Lemma J.1 (Class Comprehension)* there is a class k containing just those x in W such that no $y \leq x$ satisfies K. That is

$$x \underset{C}{\in} k \leftrightarrow W(x) \wedge (\forall y \leq x)\big(\neg K(y)\big) \qquad (J10)$$

Clearly $0 \in_C k$ since if not then it would be a $\leq$ minimal element satisfying K, hence a counterexample to equation *(J9)* . We now show that k is successor closed.

Suppose not. Then there is some $x \in_C k$ with $S(x) \notin k$. So by *(J10)* there must be some $z \leq S(x)$ with $K(z)$. However, by *Lemma J.6 (Maximal Predecessor)*, either $z \leq x$ or $z = S(x)$. But as $x \in_C k$ we can't have $z \leq x$ so $K(S(x))$.

By equation *(J9)* there must be some $y < S(x)$ with $K(y)$. But again, by *Lemma J.6 (Maximal Predecessor)*, this entails that $y \leq x$. Contradiction. Hence k is a successor closed class containing 0 and by clause *(J4)* of Δ every member of W must be an element of k. But this contradicts $(\exists x)(K(x) \wedge W(x))$.

Leaving the $\Diamond_{SC,C,S,\in_C,W,\leq,D}$ context above we may export this contradiction establishing the well-ordering property. ■

Two more lemmas are needed before we can verify the last remaining property, anti-symmetry.

Lemma J.8. $(\forall x \mid W(x))(S(x) \nleq x)$

Note that this implies W lacks a maximal element since every element in W has a successor.

Proof. By *Lemma J.1 (Class Comprehension)* let k be the class containing just those x such that $D(x) \wedge W(x) \wedge S(x) \nleq x$ It is enough to show that k is successor closed and $0 \in_C k$ since W is contained in every such class.

First, we establish that $0 \in_C k$. Suppose $S(0) \leq 0$ and consider the formula $x \neq 0$. By *Lemma J.1 (Class Comprehension)* there is some $\hat{k}$ such that $x \in_C k' \leftrightarrow x \neq 0 \wedge D(x)$. As 0 isn't a successor, $\hat{k}$ is clearly successor closed. And as $D(S(0))$ and $S(0) \neq 0$ we have $S(0) \in_C \hat{k}$. However, by *(J5)* if $S(0) \leq 0$ then, as $\hat{k}$ is successor closed, $0 \in_C \hat{k}$. This is a contradiction. Hence $0 \in_C \hat{k}$.

As every element in W is either 0 or a successor, to show that k is successor closed it is enough to show that if $S(x) \nleq x$ then $S(S(x)) \nleq x$. Suppose this fails. As $S(x) \nleq x$ there is some successor closed k' containing $S(x)$ but not x. By the same trick used above we invoke the Multiple Definitions Lemma (*Lemma H.7*) to put together applications of Simplified Choice (*Proposition 8.1)* and Simple Comprehension (*Axiom 8.4*) to infer that possibly ($\Diamond_{SC,C,S,\in_C,W,\leq,D}$) Q_x and $Q_{k'}$ select unique objects x and k' such that k' is a successor closed class and $S(x) \in_C k' \wedge \neg x \in_C k'$. We now work to transform k' into a class k'' witnessing that $S(S(c)) \nleq S(x)$.

Enter this $\Diamond_{SC,C,S,\in_C,W,\leq,D}$ context. $\mathcal{C}(C, \in_C, \text{Ext}(S))$ must remain true in this context. So by *Lemma J.1 (Class Comprehension)* there's a class k'' including every element in k' except for $S(x)$ (i.e., $k'' = k' - \{S(x)\}$). By our choice of k', k' doesn't contain x and is successor closed. So k'' must also be successor closed (for in removing $S(x)$ from k' we aren't removing the successor of anything in k'). And k'' contains $S(S(x))$, for k' contained $S(x)$ and we know $\neg S(x) = S(S(x))$ by the last clause in Ω, so k'' does as well. Thus, k'' is a successor closed class containing $S(S(x))$ but not $S(x)$. But by *(J5)* combining this with $S(S(x)) \leq S(x)$ yields contradiction, which can be exported from the above logical possibility context.

Thus, k is a successor closed class containing 0, and it follows that all x such that $W(x)$ are elements of k. Given our characterization of k, this implies that, for every x in W, $S(x) \nleq x$, as desired. ∎

Finally, we show that the definition of 0 used in this proof (the unique element that has a successor but isn't a successor) is equivalent to the definition used in the statement of the theorem (the unique $\leq$ minimal element in W). Note that is enough to prove the following lemma, saying that 0 is $\leq$ minimal, as the above lemma ensures that no other element in W is $\leq$ minimal.

Lemma J.9. $(\forall y)(y \leq 0 \rightarrow y = 0)$

Proof. Suppose y witnesses the failure of the lemma. Consider the formula $x \neq 0$. By the *Lemma J.1 (Class Comprehension)* there is some k such that $x \in_C k \leftrightarrow x \neq 0 \wedge D(x)$. As 0 isn't a successor k is clearly successor closed, and as $y \leq 0$ we have $D(y)$. So $y \in_C k$. However, by *(J5)* this contradicts the fact that $y \leq 0$ completing the proof. ∎

Lemma J.10 (Anti-symmetry). $(\forall x \mid W(x))(\forall y)(x \leq y \wedge y \leq x \rightarrow x = y)$

Proof. By the *Lemma J.1 (Class Comprehension)* let k be the class containing just those x satisfying W such that $(\forall y)(x \leq y \wedge y \leq x \rightarrow x = y)$. As above it is enough to show that $0 \in_C k$ and k is successor closed.

Note that, by the prior result, if $y \leq 0$ then $y = 0$. Hence, $0 \in_C k$.

We now establish that k is successor closed. Suppose not. Then for some x we have $x \in_C k$ but not $S(x)$. Thus, for some y, $y \leq S(x)$ and $S(x) \leq y$ but not $y = S(x)$. As $x \leq S(x)$ and $S(x) \leq y$ by transitivity we have $x \leq y$. By the *Lemma J.6 (Maximal Predecessor)* property above, since $y < S(x)$, we have $y \leq x$. As $x \in_C k$ it follows that $y = x$. But this contradicts the fact that $S(x) \nleq x$, by *Lemma J.8* above. Hence $y = S(x)$. Thus k is successor closed. ■

Note that the above lemmas verify every element of the Infinite Well-Ordering Theorem (*Theorem J.1*), completing its proof.

J.3 Possibly $\boldsymbol{PA}_\lozenge$

If follows fairly quickly from the proof above that it's logically possible that $PA_\lozenge$, where $PA_\diamond$ (given below) is the (relational) version of the second order Peano Axioms which replaces the second order induction principle with an equivalent formulation in terms of conditional logical possibility.

Definition J.1. PA$_\lozenge$ *is the formula given by the conjunction of the following clauses*

1. *The relation S is a function*[12].
2. $(\exists! z \mid \mathbb{N}(z))(\forall x \mid \mathbb{N}(x))(\neg S(x) = z \land \mathbb{N}(z))$. *As above we will refer to this unique z as 0.*
3. *(Successor Closed)* $(\forall n)[\mathbb{N}(n) \to \mathbb{N}(S(n))]$*, i.e., $\mathbb{N}$ is closed under successor.*
4. *For all natural numbers m and n, if $S(m) = S(n)$ then $m = n$. That is, S is an injection.*
5. *(Induction)* [13] $\square_{\mathbb{N},S}([(K(0) \land (\forall n \mid \mathbb{N}(n))(K(n) \to K(S(n)))] \to (\forall n \mid \mathbb{N}(n))K(n))$

Lemma J.11 (Possibly $PA_\lozenge$). *Suppose $\mathbb{N}, S$ don't appear in $\mathcal{L}$ then $\lozenge_{\mathcal{L}}(PA_\diamond)$*

That is, it's logically possible that $\mathbb{N}, S$ satisfy the Peano axioms (in the form given above) while holding fixed $\mathcal{L}$

Proof. We note that the proof of the Infinite Well-Ordering Theorem (*Theorem J.1*) can be modified to hold fixed $\mathcal{L}$ and use $\mathbb{N}$ in place of W (renaming the relations introduced in that proof as necessary to avoid collision with $\mathcal{L}$) and that in doing so all the lemmas proved in the prior section remain valid. As in the proof above we enter the $\lozenge$ (now $\lozenge_{\mathcal{L}}$) context and, assuming $\Omega \land \mathcal{C}(C, \in_C, \mathrm{Ext}(S)) \land \Delta$ derive the desired properties.

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[12] *Formally, this is the assertion that for all n, m, m', if $S(n, m)$ and $S(n, m')$ then $m' = m$. As usual we will use functional notation for S.*

[13] *Expressed in a relational form this would be* $\square_{\mathbb{N},S}[(K(0) \land (\forall n)(\forall n' \mid \mathbb{N}(n'))([K(n) \land S(n, n') \to K(n')] \to (\forall n \mid \mathbb{N}(n))K(n))]$ *but as usual we gloss over this trivial difference.*

The lemmas proved above in conjunction with Ω immediately entail all but *clause 3 (Successor Closed)* and *clause 5 (Induction)* . To prove that $\mathbb{N}$ is successor closed we note that if $\mathbb{N}(x)$ then x is in all successor closed k containing 0 and hence so is $S(x)$. Hence, $\mathbb{N}$ is successor closed.

To prove the induction claim suppose that

$$K(0) \wedge (\forall n \mid \mathbb{N}(n))(K(n) \rightarrow K(S(n)))$$

we note that by the *Lemma J.1 (Class Comprehension)* there is some class k such that

$$(\forall n) \left(n \underset{C}{\in} k \leftrightarrow \mathrm{Ext}(S)(n) \wedge K(n) \wedge \mathbb{N}(x) \right)$$

As $\mathbb{N}$ is successor closed and contains 0 as is, by assumption, K it immediately follows that $SC(k)$. Hence, it follows, by the definition of W (now $\mathbb{N}$) that if $\mathbb{N}(n)$ then $n \in_C k$ and hence $K(n)$. Thus, we can infer

$$[(K(0) \wedge (\forall n \mid \mathbb{N}(n))(K(n) \rightarrow K(S(n)))] \rightarrow (\forall n \mid \mathbb{N}(n))K(n)$$

Since we derived this conclusion from $\Omega \wedge \mathcal{C}(C, \in_C, \mathrm{Ext}(S)) \wedge \Delta$ which is content restricted to $\mathbb{N}, S, C, \in_C, D, \leq$ via *Lemma 4.3 (Box Introduction)* we can conclude

$$\Box_{\mathbb{N},S,C,\underset{C}{\in},D,\leq} ([(K(0) \wedge (\forall n \mid \mathbb{N}(n))(K(n) \rightarrow K(S(n)))] \rightarrow (\forall n \mid \mathbb{N}(n))K(n))$$

Now by *Lemma H.1 (Box Ignoring)* we can conclude.

$$\Box_{\mathbb{N},S} ([(K(0) \wedge (\forall n \mid \mathbb{N}(n))(K(n) \rightarrow K(S(n)))] \rightarrow (\forall n \mid \mathbb{N}(n))K(n))$$

Leaving the $\lozenge_{\mathcal{L}}$ context completes our proof. ■

Note that, while we don't provide a proof here, it is straightforward to add relations $+,*$ to $\mathrm{PA}_{\lozenge}$ and if we do so the above theorem continues to hold but we omit the proof of this claim.

K. Properties of Initial Segments

K.1 Isomorphism Agreement Lemmas

Informally, this lemma says that there is only one way to isomorphically map between initial segments of well-orderings.

Lemma K.1 (Well Ordering Agreement Lemma). *Suppose that* $(W, <), (W', <')$ *are well orders and*

- $(W_f, <_f), (W_g, <_g) \leq (W, <)$
- $(W'_f, <'_f), (W'_g, <'_g) \leq (W', <')$
- $(W_f, <_f) \cong_f (W'_f, <'_f)$
- $(W_g, <_g) \cong_g (W'_g, <'_g)$

then $(\forall x)\big(W_f(x) \wedge W_g(x) \rightarrow f(x) = g(x)\big)$

Proof. Suppose the assumptions in the lemma hold but that the conclusion fails and let $\mathcal{L} = \{W_f, <_f, W_g, <_g, W'_f, <'_f, W'_g, <'_g, W, <, W', <', f, g\}$. We argue that there must be a $<$ least element at which the claim fails and show that yields contradiction. By Simple Comprehension (*Axiom 8.4*) we can infer that it's logically possible ($\lozenge_{\mathcal{L}}$) that the assumptions in the lemma hold but the conclusion fails as well as

$$(\forall o)[B(o) \leftrightarrow W_f(x) \wedge W_f(x) \wedge f(o) \neq g(o)]$$

Enter this $\lozenge_{\mathcal{L}}$ context. By the definition of well ordering (*Definition E.2*), there must be some $<$ least o satisfying $B(o)$. It is trivial to verify the claim must hold if o is the $<$ least element in W.

So suppose that $f(o) \neq g(o)$ and o isn't the least element in W. Without loss of generality we may assume $f(o) <' g(o)$. By supposition $W'_g(g(o))$ and it follows by the that $W'_g(f(o))$. Thus, for some u satisfying $W_g(u)$ we have $g(u) = f(o) <' g(o)$. Now it follows that $g(u) <'_g g(o)$, hence as $(W_g, <_g) \cong_g (W'_g, <'_g)$ it follows that $u <_g o$ and thus $u < o$.

But as $W_f(o)$, we can infer (from $u < o$) that $W_f(u)$. Hence, u is in the domain of both g and f and by the minimality of o we must have $f(u) = g(u) = f(o)$, contradicting the injectivity of f.

Exporting the contradiction from the $\lozenge_{\mathcal{L}}$ context using *Axiom 8.2 (Diamond Elimination)* establishes the claim to be proved. ∎

We now prove a similar result for initial segments.

Lemma K.2 (Hierarchy Agreement Lemma). *Suppose*

- $V_f, V_g \leq V$
- $V'_f, V'_g \leq V'$
- $V_f \cong_f V'_f$
- $V_g \cong_g V'_g$

then $(\forall x)\big(V_f(x) \wedge V_g(x) \rightarrow f(x) = g(x)\big)$

Proof. Let $\mathcal{L} = \{V_f, V_g, V'_f, V'_g, V, V', f, g\}$ and suppose these relations are as in the statement of lemma but the lemma fails. tells us the claim must hold on the ordinals of the initial segments, so it remains only to must it for the sets. Our strategy here will be to prove the claim by transfinite induction on the ordinal at which x is formed and use the inductive assumption combined with extensionality to infer the claim holds for x.

By Simple Comprehension (*Axiom 8.4*) that it's possible ($\lozenge_{\mathcal{L}}$) that all the facts above continue to hold and that

$$(\forall o)\big[B(o) \leftrightarrow (\exists x \mid @(x,o))\big(set_f(x) \wedge set_g(x) \wedge \neg f(x) = g(x)\big)\big]$$

Enter this $\Diamond_{\mathcal{L}}$. By the there must be some $<$ least o satisfying $B(o)$. Let x be a set in both V_f and V_g witnessing that o satisfies B, i.e. $@(x,o)$ and $f(x) \neq g(x)$.

Now suppose $y' \in 'f(x)$. We argue that $u' \in 'g(x)$

By assumption set$'_f(f(x))$, and by the definition of Initial Segment Extension (*Definition A.3*), it follows that set$'_f(y')$. Hence by the same definition there must be some y with set$_f(y)$, $f(y) = y'$ and $y \in_f x$. And by the same definition, $y \in x$. As $@(x,o)$ ensures that there is some $o' < o$ with $@(y,o')$.

By assumption set$_g(x)$ and by the definition of Initial Segment Extension (*Definition A.3*), it follows[14] that set$_g(y)$. Hence y is in the domain of both f and g and if $f(y) \neq g(y)$ this would violate the minimality of o. As g is an isomorphism, it follows that $y' \in '_g g(x)$ and by *Definition A.3* it follows that $y' \in 'g(x)$. By a similar argument applied in the other direction we can establish that $y' \in 'f(x) \leftrightarrow y' \in 'g(x)$.

Thus, by part it follows that $f(x) = g(x)$. Contradiction[15]. Leaving the $\Diamond_{V_f,V_g,V'_f,V'_g,V,V',f,g}$ context by *Axiom 8.2 (Diamond Elimination)* we can export this contradiction giving us the desired result. ■

Corollary K.1. *If* $V_0 \leq V, \hat{V}$ *and* $V \cong_f \hat{V}$ *then* $(\forall x \mid V_0(x))(f(x) = x)$

Proof. This is immediate by taking g to be the identify function, V_g, V'_g both to be V_0, V_f to be V and V'_f to be $\hat{V}$ and applying *lemma K.2 (Hierarchy Agreement Lemma)*. ■

K.2 V Comparability Lemma

We now establish that, given any two initial segments then one extends (an isomorphic image of) the other.

Lemma K.3 (V Comparability Lemma). *If* V, V' *are initial segments then* $\Diamond_{V,V'}(\hat{V} \leq V \wedge \hat{V} \underset{f}{\cong} V') \vee (\hat{V}' \leq V' \wedge \hat{V}' \underset{f}{\cong} V)$

Proof. Our strategy here will (essentially) be to define $R(x,y)$ so that it holds just if $V(x) \wedge V'(y)$ and it is logically possible to have $V_0 \leq V, V'_0 \leq V'$ and it's logically possible for g to isomorphicly map an initial segment of V to an initial segment of V' so that $g(x) = y$. Our ultimate isomorphism f will either be defined as $f(x) = y \leftrightarrow R(x,y)$ or $f(y) = x \leftrightarrow R(x,y)$ depending on which of V or V' has the higher height.

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[14] It follows by the fact that $x \in y$ and $set_g(x)$ and $set_g(y)$ so $x \in_g y$.

[15] Note this argument applies even if $o = 0$ in which case x is the empty set.

Suppose V, V' are initial segments. As $R(x,y)$ is defined via a modal notion we must use *Axiom 8.9 (Modal Comprehension)* to define $R(x,y)$. In particular, *Axiom 8.9* allows us to infer that it's possible ($\Diamond_{V,V'}$) that V, V' are initial segments and

$$\Box_{\mathcal{L},R}\Big[\quad (\exists!\, x,y \mid Q(x,y)) \rightarrow$$
$$(\exists x,y \mid Q(x,y))[R(x,y) \leftrightarrow \mathrm{Ext}(V,V')(x) \wedge \mathrm{Ext}(V,V')(y) \wedge \phi]\Big]$$

where

$$\phi = \Diamond_{V,V',Q}\left[V_0 \leq V \wedge V'_0 \leq V' \wedge V_0 \underset{g}{\cong} V'_0 \wedge (\exists x,y)(Q(x,y) \wedge g(x) = y)\right]$$

Enter this $\Diamond_{V,V'}$ context. We will now argue that $R(x,y)$ defines the desired isomorphism.

It is evident from the definition of $R(x,y)$ that R takes sets to sets and ordinals to ordinals. We first verify that for each x there is at most one y such that $R(x,y)$. Suppose not then using the Multiple Definitions Lemma (*H 17.7*) to pack together applications of *Axiom 8.4 (Simple Comprehension)* and *Proposition 8.1 (Simplified Choice)* we can ($\Diamond_{R,V,V'}$) retain all the above facts and have Q_0 applying to a single pair x, y_0 and a Q_1 apply to a single pair x, y_1 such that $R(x,y_0) \wedge R(x,y_1)$. Entering this context and applying *Lemma B.3 (Box Elimination)* (and renaming bound first order variables) we may thus infer that both of the following hold.

$$\Diamond_{V,V',Q_0} \quad V_0 \leq V \wedge V'_0 \leq V' \wedge V_0 \underset{g_0}{\cong} V'_0 \wedge (\exists x,y_0)(Q_0(x,y_0) \wedge g_0(x) = y_0)$$
$$\Diamond_{V,V',Q_1} \quad V_1 \leq V \wedge V'_1 \leq V' \wedge V_1 \underset{g_1}{\cong} V'_1 \wedge (\exists x,y_1)(Q_1(x,y_1) \wedge g_1(x) = y_1)$$

Next, we will show that we can paste these two scenarios together. For note that the sentence inside $\Diamond_{V,V',Q_0}$ above is content restricted to V, V', Q_0, g_0, and the sentence inside $\Diamond_{V,V',Q_1}$ above is content restricted to V, V', Q_1, g_1. So the only overlap in the content of the pair of scenarios asserted to be possible above concerns relations which both of them are holding fixed (V, V'). Thus, we can apply $\Diamond$ Ignoring (*Axiom 8.3*) to get the $\Diamond_{V,V',Q_0,Q_1}$ version of both claims above, and then *Lemma B.7 (Pasting)* to infer

$$\Diamond_{V,V',Q_0,Q_1}\begin{bmatrix} V_0 \leq V \wedge V'_0 \leq V' \wedge V_0 \underset{g_0}{\cong} V'_0 \wedge (\exists x,y_0)(Q_0(x,y_0) \wedge g_0(x) = y_0) \wedge \\ V_1 \leq V \wedge V'_1 \leq V' \wedge V_1 \underset{g_1}{\cong} V'_1 \wedge (\exists x,y_1)(Q_1(x,y_1) \wedge g_1(x) = y_1) \end{bmatrix}$$

Import into this $\Diamond_{V,V',Q_0,Q_1}$ scenario the fact that $(\forall x_0)(\forall x_1)(\forall y_0)(\forall y_1)(Q_0(x_0,y_0) \wedge Q_1(x_1,y_1) \rightarrow x_0 = x_1 \wedge y_0 \neq y_1)$. Thus, the scenario under the $\Diamond_{V,V',Q_0,Q_1}$ one in which g_1 and g_0 isomorphicly map the initial segments $V_0, V_1 \leq V$ to $V'_0, V'_1 \leq V'$. However, this scenario is exactly what is ruled out by the *Lemma K.2 (Hierarchy Agreement)* giving us a contradiction which can be exported to infer that $R(x,y)$ is injective. The fact that R is an injective function justifies are use of functional notation (e.g., $R(x) = y$) for the remainder of the proof. And since, if g_0, g_1 are isomorphisms so are g_0^{-1}, g_1^{-1}, the same considerations above imply that if $R(x,y)$ and $R(x',y)$ then $x = x'$.

We now argue that if x_0, x_1 are sets in V and both $R(x_0)$ and $R(x_1)$ are defined then $x_0 \in x_1 \leftrightarrow R(x_0) \in R(x_1)$. First, assume that there is some x_0, x_1 with $x_0 \in x_1$ but $R(x_0) \notin R(x_1)$. By the same argument above (building Q_0, Q_1 applying to $(x_0, R(x_0))$ and $(x_1, R(x_1))$ respectively and then applying *Lemma B.7 (Pasting)*), we may assume we are in a context in which we have isomorphisms $g_0(x_0) = R(x_0)$ and $g_1(x_1) = R(x_1)$ and, since $x_0 \in x_1$ and thus in the domain of g_1. Applying this the *Lemma K.2 (Hierarchy Agreement)* in this context we may conclude that $g_1(x_0) = g_0(x_0)$ from which the conclusion $R(x_0) \in R(x_1)$ follows. This yields the desired contradiction which we can export to the original context giving us that $R(x_0) \in R(x_1)$ in that context. A similar argument lets us infer that if $R(x_0) \in R(x_1)$ then $x_0 \in x_1$. This is enough to show that R respects $\in$. Similar reasoning demonstrates that R respects $<$ and $@$.

We now argue that the domain and range of R are initial segments of V, V' respectively. We note that if x is a set in V and x is in the domain of R then it's possible (speaking loosely) that x in the domain (range) of g and g isomorphically maps some $V_0 \leq V$ to ${V'}_0 \leq V$ then, since x must be available at some ordinal u in V_0 (${V'}_0$) it follows that x is available at some ordinal in the domain (range) of R. Similarly, if o is an ordinal in the domain (range) of R and $u < o$ then u is in the domain (range) of R. Thus, by *Lemma E.1 (Initial Segment)* we can infer that the domain of R is some initial segment $\hat{V} \leq V$ and the range is some initial segment $\hat{V}' \leq V'$ and $\hat{V} \cong_R \hat{V}'$.

Finally, it remains to show that either $\hat{V} = V$ or $\hat{V}' = V'$ Suppose not. Then $\hat{V} < V$ and $\hat{V}' < V'$. Since $\hat{V}$ is an initial segment there must be some ordinal o in V not in the domain of R. We now use *Axiom 8.4 (Simple Comprehension)* via *Lemma H.7 (Multiple Definitions)* to define $B(o)$ to hold on just those ordinals in V not in the domain of R and B' to hold on those ordinals of V' not in the range of R (by assumption both of which are non-empty).

Again using Simple Comprehension (*Axiom 8.4*) (replacing o, o' with their definition in terms of B, B') we infer the possibility of a relation g defined on ordinals $u \leq o$ in V by

$$g(u) = \begin{cases} R(u) & \text{if } u < o \\ o' & \text{if } u = 0 \end{cases}$$

and for sets x in V with $@(x, o)$ we define

$$g(x) = \begin{cases} R(x) & \text{if dom}(R)(x) \\ y & \text{otherwise, where } y' \in y \leftrightarrow (\exists x' \mid x' \in x)(y' = R(x)) \end{cases}$$

Note that the existence of such a y is guaranteed by the fact that $\hat{V}' \neq V'$ and the fatness requirement on V'. With this construction in hand we can straightforwardly verify that g is an isomorphism and, importing the definition of R, conclude that $R(o, o')$ contradicting the fact that $B(o)$ and $B'(o')$. Exporting this contradiction we conclude that either $\hat{V} = V$ or $\hat{V}' = V'$.

We now use Simple Comprehension (*Axiom 8.4*) (inside the initial $\Diamond_{V,V'}$ context) to show that possibly $\Diamond_{V,V',R}$ define f so that, if $\hat{V}' = V'$, then $f(x) = y \leftrightarrow R(x, y)$ and if $\hat{V} = V$ then $f(y) = x \leftrightarrow R(x, y)$. As R was already shown to be an isomorphism between the initial segments $\hat{V}$ and $\hat{V}'$ we may complete the proof by applying Diamond Collapsing (*Lemma B.4*). ■

K.3 Proper Extension Lemma

Lemma K.4 (Proper Extension Lemma). *If V is an initial segment, then $\Diamond_V\,(V' \geq V)(\exists o)(\mathrm{ord}'(o) \land (\forall u)(\mathrm{ord}(u) \to u <' o))$*

Proof. Our strategy here will be to invoke the possibility of a layer of classes over the elements satisfying set. We will then take the set′ to include all the objects satisfying set together with those of these classes which can't be identified with existing sets, with membership defined in the obvious fashion. We will extend the ordinals in V by adding a single new object (the empty class) which is an ord' but not an ord.

More formally, suppose V is an initial segment and use Possible Powerset (*Axiom 8.11*) to infer that $\Diamond_{\mathrm{set}}\,\mathcal{C}\,(C, \in_C, \mathrm{set})$. As $\mathcal{C}(C, \in_C, \mathrm{set})$. is content restricted to $C, \in_C$,set we can apply $\Diamond$ Ignoring (*Axiom 8.3*) to expand the list of relations held fixed to V, i.e., set,ord, @. Additionally we can use the Multiple Definitions Lemma (*Lemma H.7*) to infer the logical possibility ($\Diamond_{V,C,\in_C}$) that each definition in the chain of definitions below holds along with the facts above ($\mathcal{C}(C, \in_C, \mathrm{set})$ and $\mathcal{V}(V)$).

$$(\forall x)[\mathrm{set}'(x) \leftrightarrow (\mathrm{set}(x) \lor [C(x) \land (\forall y)(\mathrm{set}(y) \to (\exists z)\neg(z \in y \leftrightarrow z \underset{C}{\in} x))]))$$
$$(\forall x)(\forall y)[x \in' y \leftrightarrow \mathrm{set}'(x) \land \mathrm{set}'(y) \land (x \in y \lor x \underset{C}{\in} y)]$$
$$(\forall x)[\mathrm{ord}'(x) \leftrightarrow (\mathrm{ord}(x) \lor (C(x) \land (\forall y)(\neg y \underset{C}{\in} x)))]$$
$$(\forall x)(\forall y)[x \leq' y \leftrightarrow (\mathrm{ord}'(x) \land \mathrm{ord}'(y) \land (x < y \lor (\mathrm{ord}(x) \land \neg\mathrm{ord}(y))))]$$
$$(\forall x)(\forall y)[@'(x, y) \leftrightarrow (\mathrm{set}(x)' \land \mathrm{ord}'(y) \land (@(x, y) \lor \mathrm{set}(x) \land \neg\mathrm{ord}(y)))]$$

From these definitions it is straightforward, if tedious, to verify the claimed result. The only significant departure from familiar first order reasoning concerns showing that our new V' obeys part 5 (fatness) from the definition of Initial Segment (*Definition A.2*). Suppose, for contradiction, that fatness fails. Then possibly ($\Diamond_{V'}$) H applies to some sets in V' all of which are available before some o satisfying $\mathrm{ord}'(o)$ but that no set whose members are equal to H is available at stage o. By $\Diamond$ Ignoring (*Axiom 8.3*) we can infer the $\Diamond_{V,V',C,\in_C}$ version of this claim. Enter this $\Diamond_{V,C,\in_C}$ context. By the facts about content restriction labeled above, we can import all our sentences characterizing $C, \in_c$,set, $\in$ etc. into this context and derive that H applies to only elements in V (since V' adds only a single new ordinal) and thus there is a class x whose members are exactly the sets y satisfying H and thus $\mathrm{set}'(x)$. If $o < \emptyset_C$ (the new ordinal in V') then $@(x, o)$ and thus $@'(x, o)$. If $o = \emptyset_C$ then $\neg\mathrm{ord}(o)$ and thus $@'(x, o)$. This gives us the desired contradiction which we can export. ■

Lemma K.5 (Interpreted Initial Segment Possibility). *Suppose that $\mathcal{L}$ doesn't contain any of* set,ord, $\in, <, @, \mathbb{N}, S, \rho$ *then* $\Diamond_{\mathcal{L}}\,\mathcal{V}\,(\vec{V})$

Proof. By Possibly $PA_\Diamond$ (*Lemma J.11*) we may infer that $\Diamond_{\mathcal{L}}\,\mathrm{PA}_\Diamond$ Enter this $\Diamond_{\mathcal{L}}$ context.

We note that by the Multiple Definitions Lemma (*Lemma H.7*) and Simple Comprehension (*Axiom 8.4*) (letting all relations be empty) we can trivially deduce the possibility of an (empty) initial segment, i.e., $\Diamond_{\mathcal{L}}\, \mathrm{PA}_{\diamond} \wedge \mathcal{V}(V)$. Enter this $\Diamond_{\mathcal{L}}$ context and apply the Proper Extension Lemma (*Lemma K.4*) to infer

$$\Diamond_{\mathrm{V},\mathcal{L}}\ (V' \geq V) \wedge \mathrm{PA}_{\diamond} \wedge (\exists o)\mathrm{ord}'(o)$$

Enter the above $\Diamond_{V,\mathcal{L}}$ context. It is easy to verify (using the) that there is a unique object satisfying set' which has no members. Using Simple Comprehension (*Axiom 8.4*) we may define $\rho(n)$ for all n satisfying $\mathbb{N}(n)$ to be this empty set in V'. It is easy to verify that this entails that V' is an interpreted initial segment (*Definition A.4*).

Leaving all the above $\Diamond$ contexts we may use Diamond Collapsing (*Lemma B.8*) and Relabeling (*Axiom 8.5*) to infer $\diamond\vec{\mathcal{V}}(\vec{V})$ as desired. ■

Corollary K.2 (Interpreted Extension). *Suppose that $\mathcal{L}$ contains no relations in $\vec{V}$ then $\mathcal{V} \wedge (\exists x)(\mathrm{set}(x)) \rightarrow \Diamond_{V,\mathcal{L}}\vec{\mathcal{V}}(\vec{V})$*

Proof. This is immediate via the proof of Interpreted Initial Segment Possibility (*Lemma K.5*) by substituting the assumption that $\mathcal{V} \wedge (\exists x)(\mathrm{set}(x))$ in place of the construction of V' in that proof. ■

Lemma K.6 (Proper Well-Ordering Extension Lemma). *If* $\mathrm{ord}, <$ *is an initial segment then* $\Diamond_{\mathrm{ord},<} ((\mathrm{ord}', <') \geq \mathrm{ord}, <))(\exists o)(\mathrm{ord}'(o) \wedge (\forall u)(\mathrm{ord}(u) \rightarrow u <' o))$ *Moreover, we may assume that* $\mathrm{ord}', <'$ *has a maximal element, i.e.,* $\Diamond_{\mathrm{ord},<} ((\mathrm{ord}', <') \geq \mathrm{ord}, <))(\exists o)(\mathrm{ord}'(o) \wedge \neg\mathrm{ord}(o) \wedge (\forall u \mid \mathrm{ord}'(u))(u \leq' o))$

Proof. By the same reasoning as in the proof of the Proper Extension Lemma (*Lemma K.4*) regarding the ordinals. ■

K.4 Hierarchy Extending Lemma

Lemma K.7 (Hierarchy Extending Lemma). *If $\mathcal{V}(V_a) \wedge \mathcal{V}(V_b)$ then $\Diamond_{V_a,V_b} V^+ \geq V_a \wedge V^+ \geq V_b' \wedge V_b' \underset{f}{\cong} V_b$ Moreover, assuming V^+, V'_b don't occur in $\mathcal{L}$ we may also infer $\Diamond_{V_a,V_b,\mathcal{L}} V^+ \geq V_a \wedge V^+ \geq V_b' \wedge V_b' \underset{f}{\cong} V_b \wedge (\forall x \mid \mathrm{Ext}(\mathcal{L})(x))(V^+(x) \rightarrow V_a(x))$*

This lemma tells us that it's logically possible to find a common extension[16] for any two initial segments and, specifically, that we can take it to extend V_a The moreover claim ensures that the new elements in this common extension can be taken not to overlap with those in the extension of any given list of relations $\mathrm{Ext}(\mathcal{L})$.

Proof. We first prove the main claim. By the V Comparability Lemma (*Lemma K.3*) we have

$$\Diamond_{V_a,V_b} (V_a \geq \hat{V}' \wedge \hat{V}' \underset{f}{\cong} V_b) \vee (V_b \geq \hat{V} \wedge \hat{V} \underset{f}{\cong} V_a)$$

[16] More precisely, an extension of some isomorphic image.

Enter this $\Diamond_{V_a, V_b}$ context. We note that it is enough to deduce

$$\Diamond_{V_a, V_b} V^+ \geq V_a \wedge V^+ \geq V_b{}' \wedge V_b{}' \cong V_b$$

in this context as it can be exported to prove the desired conclusion.

If the first disjunct holds (i.e., V_a is the taller initial segment), the claim follows almost immediately by letting V^+ be V_a. Specifically, we use Simple Comprehension (*Axiom 8.4*) to establish the possibility ($\Diamond_{V_a, V_b, f, \hat{V}'}$) that $V^+ = V_a, V'_b = \hat{V}'$ while maintaining all relevant facts[17]. Then, entering this $\Diamond$ context we may derive that $V^+ \geq V_a \wedge V^+ \geq V_b{}' \wedge V_b{}' \cong_f V_b$. We can then leave this context giving us

$$\Diamond_{V_a, V_b, f, \hat{V}'} V^+ \geq V_a \wedge V^+ \geq V_b{}' \wedge V_b{}' \cong V_b$$

By Reducing (*Lemma 4.1*) we can infer

$$\Diamond_{V_a, V_b} V^+ \geq V_a \wedge V^+ \geq V_b{}' \wedge V_b{}' \cong V_b$$

which, as we noted above, suffices to prove the lemma.

So we may instead assume

$$\hat{V} \leq V_b \wedge \hat{V} \underset{f}{\cong} V_a \qquad (K1)$$

In the special case where V_a and V_b are disjoint it is easy to see how to proceed. We simply define V^+ to supplement V_a with all elements in V_b not in V_a using f to define $\in_+$ and $<_+$ so that V^+ treats elements in V_a identically to their isomorphic images in V_b. We can then define f' to extend f by being the identity on those elements in V^+ but not V_a. More formally, we can use the the Multiple Definitions Lemma (*Lemma H.7*) with Simple Comprehension (*Axiom 8.4*) to show it's possible ($\Diamond_{V_a, V_b, f, \hat{V}}$) that all facts from the current context remain true and

$$\begin{aligned}
(\forall x)(\text{set}^+(x) \leftrightarrow \quad & (\text{set}_a(x) \vee (\text{set}_b(x) \wedge \neg(\exists z \mid \text{set}_a(z))(f(z) = x))) \\
(\forall o)(\text{ord}^+(o) \leftrightarrow \quad & (\text{ord}_a(o) \vee (\text{ord}_b(o) \wedge \neg(\exists u \mid \text{ord}_a(u))(f(u) = o))) \\
(\forall x)(\forall y)(f'(x) = y \leftrightarrow \quad & (f(x) = y \vee (x = y \wedge [V_+(x) \wedge \neg V_a(x)])) \\
(\forall x)(\forall y)[x \underset{+}{\in} y \leftrightarrow \quad & (\exists x')(\exists y')(f'(x) = x' \wedge f'(y) = y' \wedge x' \underset{b}{\in} y')] \\
(\forall x)(\forall y)[x \underset{+}{<} y \leftrightarrow \quad & (\exists x')(\exists y')(f'(x) = x' \wedge f'(y) = y' \wedge x' \underset{b}{<} y')] \\
(\forall x)(\forall y)[@_+(x, y) \leftrightarrow \quad & (\exists x')(\exists y')(f'(x) = x' \wedge f'(y) = y' \wedge @_b(x', y'))]. \qquad (K2)
\end{aligned}$$

Since we've explicitly defined V^+ to extend V_a by copying V_b and f' to extend f by defining it to be the identify on the part of V_b copied to V^+ it is straightforward to check that f' is the desired isomorphism giving us

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[17] Specifically, we may take the conjunction of all sentences true in the current context and conjoin them with the definition given by *Axiom 8.4 (Simple Comprehension)* under $\Diamond_{V_a, V_b, f, \hat{V}'}$.

$$\Diamond_{V_a,V_b,f,\hat{V}}\, V^+ \geq V_a \wedge V^+ \geq V_b{}' \wedge V_b{}' \underset{f}{\cong} V_b$$

which we may again apply Reducing (*Lemma B.4*) to infer (sufficient by the remarks above)

$$\Diamond_{V_a,V_b}\, V^+ \geq V_a \wedge V^+ \geq V_b{}' \wedge V_b{}' \cong V_b$$

In the general case where V_a and V_b may overlap, we pursue the same strategy but invoke the Possible Powerset axiom (*Axiom 8.11*) with respect to those objects in either V_a or V_b, to get the logical possibility ($\Diamond_{V_a,V_b}$) of having a layer of classes over the V_a, V_b structure, disjoint from all objects in $\text{Ext}(V_a, V_b)$. Then, instead of using elements from V_b to extend V_a, we use a possible $V_b{}'$ whose elements are all the singleton classes of elements from V_b.

Specifically, by Possible Powerset (*Axiom 8.11*) we may derive

$$\Diamond_{V_a,V_b,f,\hat{V}}\, \mathcal{C}\,(C, \underset{C}{\in}, \text{Ext}(V_a, V_b))$$

Entering this $\Diamond_{V_a,V_b,f,\hat{V}}$, context we import *(K1)* as well as the assumptions of the lemma and note that by *Lemma H.8 (Singleton)* we may assume that there is a unique singleton associated with every element satisfying $\text{Ext}(V_a, V_b)$. Adopting the abbreviation $y = \{x\}_C \wedge \Psi$ for the claim that $(\exists y \mid C(y))[(\forall x')(x' \in_C y \leftrightarrow x' = x) \wedge \Psi]$ and invoking *Lemma H.7 (Multiple Definitions)* we can define

$$\begin{aligned}
(\forall x)(\text{set}^+(x) \leftrightarrow \quad & (\text{set}_a(x) \vee x = \{y\} \wedge (\text{set}_b(y) \wedge \neg(\exists z \mid \text{set}_a(z))(f(z) = y))) \\
(\forall o)(\text{ord}^+(o) \leftrightarrow \quad & (\text{ord}_a(o) \vee o = \{u\} \wedge (\text{ord}_b(u) \wedge \neg(\exists u' \mid \text{ord}_a(u'))(f(u') = u))) \\
(\forall x)(\forall y)(f'(x) = y \leftrightarrow \quad & (f(x) = y \vee (x = \{y\} \wedge [V_+(x) \wedge \neg V_a(x)]))
\end{aligned}$$

repeating the definitions of $\in_+, <_+$ and $@_+$ just as they are in equation *(K2)*. Note that since the singletons used in the above definitions are an exact copy of V_b but guaranteed to be disjoint from V_a by *Axiom 8.11 (Possible Powerset)* we may verify this entails the desired conclusion just as in the above case.

The moreover claim follows by the same reasoning as above, but using $\Diamond$ Ignoring (*Axiom 8.3*) to add $\mathcal{L}$ to the subscript of the conclusion of the the V Comparability Lemma (*K 20.3*) and then propagating it through the remainder of the proof. The only other modification that is necessary is to invoke Possible Powerset (*Axiom 8.11*) with $\text{Ext}(V_a, V_b, \mathcal{L})$ rather than just $\text{Ext}(V_a, V_b)$ to ensure elements we use to extend V_a to V^+ can't be in $\text{Ext}(\mathcal{L})$. ■

K.5 Hierarchy-Combining

We now prove that given any indexed collection of initial segments V_x (for x satisfying $I(x)$) it is possible to find a single initial segment V_Σ which extends (an isomorphic copy of) each V_x.

Theorem K.1 (Hierarchy-Combining Theorem). *Suppose that for each x satisfying $I(x)$ V_x is an initial segment, i.e.,*

$$\Box_{\mathcal{L},I,V}\,[(\exists!\, x \mid Q(x))(I(x) \wedge \Upsilon(x)) \rightarrow \mathcal{V}(V_*))]$$

where

$$\begin{aligned}\varUpsilon(x) = \quad & (\forall z)(\mathrm{set}_*(z) \leftrightarrow \mathrm{set}(z,x)) \\ & (\forall z)(\mathrm{ord}_*(z) \leftrightarrow \mathrm{ord}(z,x)) \\ & (\forall z,y)\left(z \underset{*}{\in} y \leftrightarrow \in (z,y,x)\right) \wedge \\ & (\forall o,u)\left(o \underset{*}{<} u \leftrightarrow < (o,u,x)\right) \wedge \\ & (\forall o,z)(@_*(z,o) \leftrightarrow @(z,o,x))\end{aligned}$$

Then it is logically possible that some initial segment V_Σ extends (an isomorphic copy of) each V_x, i.e., Or, more formally

$$\begin{aligned}\Diamond_{\mathcal{L},V,I}[\quad & \vec{\mathcal{V}}(V_\Sigma) \wedge \mathrm{rng}(f) \subseteq \mathrm{Ext}(V_\Sigma) \wedge \underline{v} \\ & \Box_{\mathcal{L},V,I,f}\left((\exists! x \mid Q(x))(I(x) \wedge \varUpsilon(x)) \to \Diamond_{\mathcal{L},\hat{V},I,f,V_*}\left(V_* \underset{f}{\cong} V^- \wedge V^- \leq V_\Sigma\right)\right)]\end{aligned}$$

In the following proof we will replace V_* with the suggestive notation of V_x for the initial segment V_* on the assumption that $\varUpsilon(x)$ holds, i.e., initial segment formed by slotting into the final position of the relations in V (aside from $\mathbb{N}, S$). While this is an abuse of notation we believe the suggestive notation makes the proof easier to understand.

Proof. We give the high-level argument used in this proof leaving the, now familiar, low level details of entering and leaving $\Diamond$ contexts to the reader.

First, we demonstrate that it suffices to prove the claim under the assumption that for $x \neq y$ the structures V_x and V_y are disjoint, i.e., . $V(z,x) \wedge V(z,y) \to x = y$, For, if not, we can borrow the trick from *Lemma K.7 (Hierarchy Extending)* of using Possible Powerset (*Axiom 8.11*) to provide us with appropriately related new versions of the structures. In particular, if V doesn't satisfy this disjointness criteria we may generate a V' which does by applying Possible Powerset (*Axiom 8.11*) 2 times[18] and defining (using (x,y) to abbreviate $\{\{x\},\{x,y\}\}$ as defined by our applications of Possible Powerset (*Axiom 8.11*).

$$\begin{aligned}\mathrm{set}'(z,x) &\leftrightarrow (\exists y)(z = (x,y) \wedge \mathrm{set}(y,x)) \\ \mathrm{ord}'(z,x) &\leftrightarrow (\exists y)(z = (x,y) \wedge \mathrm{ord}(y,x)) \\ <'(z_0,z_1,x) &\leftrightarrow (\exists y_0,y_1)(z_0 = \{(x,y_0) \wedge < (y_0,y_1,x)) \\ \in'(z_0,z_1,x) &\leftrightarrow (\exists y_0,y_1)(z_0 = (x,y_0) \wedge \in (y_0,y_1,x)) \\ @'(z_0,z_1,x) &\leftrightarrow (\exists y_0,y_1)(z_0 = (x,y_0) \wedge @(y_0,y_1,x))\end{aligned}$$

It is straightforward to check that V_x and V'_x are isomorphic and the disjointness properties of Possible Powerset (*Axiom 8.11*) guarantee they are pairwise disjoint. Since the conclusion of the lemma only depends on the isomorphism classes of the structures V_x proving the result for V' is enough to establish it for V. Thus, we may assume that V_x and V_y are disjoint for $x \neq y$.

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[18] Here we identify all classes of elements from $\mathrm{Ext}(V)$ with the classes introduced by the first application of Possible Powerset (*Axiom 8.11*) and any class containing a class from the first application with the second application of Possible Powerset/

With this assumption in place we now proceed to construct V_Σ. Our strategy here will be to build V_Σ out of the *equivalence classes* induced on elements of V (i.e. $\{y \mid (\exists x \mid I(x))(V(y,x))\}$) by the relation of isomorphic image. In other words we define $y \sim z$ to hold just if

$$
\begin{aligned}
(\forall z)(\forall y)[z \sim y \leftrightarrow \quad & \big(\exists x_z, x_y \mid I(x_z) \wedge I(x_y)\big) \Diamond_{\hat{V}, V_{x_z}, V_{x_y}, z, y} (\\
& V'_{x_z} \leq V_{x_z} \wedge V'_{x_y} \leq V_{x_y} \wedge V'_{x_z} \underset{h}{\cong} V'_{x_y} \wedge h(z) = y)]
\end{aligned}
$$

Where the use of x_z, x_y in the subscript of the $\Diamond$ indicates that we define $\sim$ using *Axiom 8.9*[19] I will use this notation without further comment in the rest of the proof.

We observe that $\sim$ forms an equivalence relation. Clearly if $\text{Ext}(V)(z)$ then $z \sim z$ (if $V_x(z)$ then[20] we let, via Simple Comprehension (*Axiom 8.4*), f be the identity and $V_{x_z} = V_{x_z} = V_x$) so reflexivity is satisfied. Since, if f is an isomorphism so is f^{-1} it is easy to see $\sim$ is symmetric. Now for transitivity suppose that $z \sim y$ and $y \sim w$. By *Lemma B.7 (Pasting)* and Relabeling (*Axiom 8.5*) we can establish[21] the simultaneous logical possibility of an isomorphism f from $V'_{x_z} \leq V_{x_z}$ to $V'_{x_y} \leq V_{x_y}$ and an isomorphism g from $V^*_{x_y} \leq V_{x_y}$ to $V^*_{x_w} \leq V_{x_w}$. We now argue that we can compose f and g or f^{-1} and g^{-1} to demonstrate $z \sim w$. The only difficulty here is to ensure that this composition has appropriate ranges and domains.

By the V Comparability Lemma (*Lemma K.3*) we can also assume that either $V^*_{x_y}$ extends an isomorphic image of V'_{x_y} or vice versa. By *Lemma K.2* applied to this isomorphism and the identity it follows that either $V^*_{x_y} \leq V'_{x_y}$ or $V'_{x_y} \leq V^*_{x_y}$. In the later case $g \circ f$ witnesses that $z \sim w$ and in the former case $f^{-1} \circ g^{-1}$ witnesses that $w \sim z$. This suffices to demonstrate $\sim$ is an equivalence relation.

To build V_Σ we need to have a single object corresponding to each equivalence class. We do this by applying Possible Powerset (*Axiom 8.11*) to add a layer of classes and identifying each equivalence class with the class of its elements. We denote the class consisting of all y such that $x \sim y$ by $[x]$. We now use these equivalence classes to define, via application of Simple

---

[19] In particular, modal comprehension guarantees it's possible that $\sim$ applies so that, necessarily (holding fixed $\sim$) if Q applies to the unique pair y, z then $y \sim z$ just if the above formula holds with all mentions of x and y replaced with their definition in terms of Q.

[20] To spell this out formally we'd need to invoke *Proposition 8.1 (Simplified Choice)* and Simple Comprehension (*Axiom 8.4*) to build Q applying to a unique z witnessing failure and then Q' to a unique x with $V_x(z)$ but we omit these now familiar details.

[21] To provide a formal proof we'd need to invoke *Proposition 8.1 (Simplified Choice)* and Simple Comprehension (*Axiom 8.4*) in the usual manner to generate Q_0, Q_1 applying to unique pairs z, y and y, w that witness this failure and use the modal definition of $\sim$ to find initial segments which witness this failure before invoking pasting.

Comprehension (*Axiom 8.4*) V_Σ (below bold faced variables are taken to range over such equivalence classes).

$$\begin{aligned}
\mathrm{set}_\Sigma(\mathbf{z}) \quad &\leftrightarrow (\exists z)(\exists x \mid I(x))(\mathrm{set}_x(z) \wedge \mathbf{z} = [z]) \\
\mathbf{z} \underset{\Sigma}{\in} \mathbf{y} \quad &\leftrightarrow (\exists z, y)(\exists x \mid I(x))\left(V_x(z) \wedge V_x(y) \wedge \mathbf{z} = [z] \wedge \mathbf{y} = [y] \wedge z \underset{x}{\in} y\right) \\
\mathrm{ord}_\Sigma(\mathbf{u}) \quad &\leftrightarrow (\exists u)(\exists x \mid I(x))(\mathrm{ord}_x(u) \wedge \mathbf{u} = [u]) \\
\mathbf{u} \underset{\Sigma}{<} \mathbf{o} \quad &\leftrightarrow (\exists u, o)(\exists x \mid I(x))\left(V_x(o) \wedge V_x(u) \wedge \mathbf{u} = [u] \wedge \mathbf{o} = [o] \wedge u \underset{x}{<} o\right) \\
@(\mathbf{z}, \mathbf{o}) \quad &\leftrightarrow (\exists z, o)(\exists x \mid I(x))(V_x(z) \wedge V_x(o) \wedge \mathbf{o} = [o] \wedge \mathbf{z} = [z] \wedge @_x(z, o))
\end{aligned}$$

We now observe that the map $[\cdot]$ taking elements to their equivalence classes is an isomorphism of V_x with some $\hat{V}_x \leq V_\Sigma$. It is already apparent that $[\cdot]$ respects $\in_x, <_x, @_x, \mathrm{set}_x, \mathrm{ord}_x$. To see that $[\cdot]$ is injective on V_x note that if $V_x(z)$ and $V_x(y)$ and $[z] = [y]$ we would need[22] V^*, V' with $V^* \cong_h V'$ where $h(z) = y$. But by our assumption of disjointness we have $V^*, V' \leq V_x$ and by the *Lemma K.2* h must be the identity so $z = y$.

It remains to show that V_Σ is an initial segment and if $\hat{V}_x$ is the image of V_x under $[\cdot]$ then $\hat{V}_x \leq V_\Sigma$. To this end we note that for any x, x' satisfying I we have $\hat{V}_x \leq \hat{V}_{x'}$ or $\hat{V}_{x'} \leq \hat{V}_x$. This follows since, by the the V Comparability Lemma (*Lemma K.3*), it's possible that either $V_x \cong_f V^* \leq V_{x'}$ or $V_{x'} \cong_f V \leq V_x$. Without loss of generality assume we are in the former case. Then, by the definition of $\sim$, it follows that if $V_x(y)$ then $[y] = [f(y)]$. So the image of V_x under $[\cdot]$ is equal to the image of V^* under $[\cdot]$. And as $V^* \leq V_{x'}$, it follows that $\hat{V}_x \leq \hat{V}_{x'}$.

Thus, V_Σ is the 'union' of a sequence of compatible initial segments. It is straightforward, if tedious, to verify that V_Σ is an initial segment using this observation. For instance, to verify that V_Σ satisfies Fatness () we note that for any $\mathbf{o}$ satisfying $\mathrm{ord}_\Sigma(\mathbf{o})$ there is an x satisfying $I(x)$ and an o satisfying $\mathrm{ord}_x(o)$ with $\mathbf{o} = [o]$ and we invoke fatness in V_x to verify fatness in V_Σ. Since all the conditions in the definition of initial segment (*Definition A.2)* are closure conditions, taking the union of compatible structures (the initial segments $\hat{V}_x$) must also satisfy these conditions. Once this is verified it is also clear that $\hat{V}_x \leq V_\Sigma$ and that by using f instead of $[\cdot]$ we've established the claim to be proved. Of course, a formal proof requires more careful attention to the $\Box$ and $\Diamond$ contexts but that manipulation should be familiar by now. ∎

We can also prove a corollary (which will be crucial for justifying replacement) which says that if there's some common V_0 such that for all of the V_xs we have $V_0 \leq V_x$, we may assume $V_0 \leq V_\Sigma$.

Corollary 20.3. *Suppose that V_0 is an initial segment and $\mathcal{L}, V, I$ satisfy the conditions of* Theorem K.1 *with the additional assumption that $V_x \geq V_0$ for each x, i.e.,*
$\Box_{\mathcal{L},I,V,V_0}[(\exists! x \mid Q(x))(I(x) \wedge Y(x)) \rightarrow V_* \geq V_0)]$

[22] Again, a full proof of this claim would require defining relations that apply to a unique tuple witnesses of the failure of the claim and then applying *Lemma B.7 (Pasting)* to establish the simultaneous logical possibility of V^* and V'.

then we may take V_Σ to extend V_0, i.e.,

$$\Diamond_{\mathcal{L},V,I}\,[\quad \mathcal{V}(V_\Sigma) \wedge V_\Sigma \geq V_0 \wedge \text{rng}(f) \subseteq \text{Ext}(V_\Sigma) \wedge$$
$$\Box_{\mathcal{L},V,I,f}\,(\exists!\,x \mid Q(x))(I(x) \wedge \Upsilon(x)) \rightarrow \Diamond_{\mathcal{L},\hat{V},I,f,V_*}\,(V_* \underset{f}{\cong} V_- \wedge V_- \leq V_\Sigma)]$$

Proof. This follows by the same argument used in *Theorem K.1 (Hierarchy Combining)* excepting only that we replace the assumption of disjointness with disjointness modulo V_0 (and using the *Lemma K.2* on V_0 to infer injectivity of $[\cdot]$) and then replace the equivalence class containing an element x from V_0 with x itself. We leave the details of this proof to the reader. ■

We can also derive a corollary which says that if the ordinals of each V_x form an initial segment of $W, <$ then we can take the ordinals in V_Σ to be as well.

Corollary 20.4. *Suppose that $\mathcal{L}, I, V$ are as in the* Theorem K.1 (Hierarchy Combining) *and that each for each x satisfying I,* $(\text{ord}_x, <_x) \leq (W, <)$ *then the conclusion of* Theorem K.1 *holds where*[23] *V_Σ satisfies* $(\text{ord}_\Sigma, <_\Sigma) \leq (W, <)$

Proof. This proof proceeds just as the proof of *Corollary K.3 (to Hierarchy Combining)* above, except instead of insisting the initial segments V_x are replaced with initial segments extending V_0 and are otherwise disjoint here we replace the initial segments V_x with initial segments whose ordinals are all drawn from compatible well-orders. We replace the singleton equivalence classes forming the ordinals of V_Σ with their unique element. ■

K.6 Fleshing Out

Theorem K.2 (Fleshing Out Theorem). *If* $\text{ord}, <$ *is a well-order then* $\Diamond_{\text{ord},<} \mathcal{V}(V)$ *where* $V = (\text{ord}, <, \text{set}, \in, @)$.

Proof. Assume that $\text{ord}, <$ is a well-order. We first note that without loss of generality we may assume that $\text{ord}, <$ has a maximal element. For, by *Lemma K.6 (Proper Well Ordering Extendability)* we may derive the possibility ($\Diamond_{\text{ord},<}$) of a well-order $\text{ord}', <'$ extending $\text{ord}, <$ with a maximal element. If we can now derive $\Diamond_{\text{ord}',<'} \mathcal{V}(V')$ then, we may invoke $\Diamond$ Ignoring (*Axiom 8.3*) to derive $\Diamond_{\text{ord}',<',\text{ord},<} \mathcal{V}(V')$ and import the fact that $(\text{ord}, <) \leq (\text{ord}', \leq')$ to derive

$$\Diamond_{\text{ord},<} \Diamond_{\text{ord},<,\text{ord}',<'} \, (\text{ord}, <) \leq (\text{ord}', <') \wedge \mathcal{V}(V')$$

From here it is easy to take the restriction of V to just those sets available at stages in ord and then by Diamond Collapsing (*Lemma B.4*) derive

$$\Diamond_{\text{ord},<} \mathcal{V}(V)$$

So assuming that $\text{ord}, <$ has a maximal element, define B so that $B(u)$ holds just if it's possible to have an initial segment of the sets whose ordinals are an initial segment of $W, <$ and contain u. Specifically using *Axiom 8.9* Modal Comprehension (using *Axiom 8.7* Logical Closure and

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[23] *That is we can add this as a conjunct under the $\Diamond_{I,V,\mathcal{L}}$ operator.*

Axiom 8.5 Relabeling to simplify the result) we infer that it's possible ($\Diamond_{\text{ord},<}$) for ord, $<$ to be a well-order with maximal element and

$$\Box_{\text{ord},<,B}\Big(\quad (\exists!\, x \mid H(x)) \to (\exists x \mid H(x))[B(x) \leftrightarrow \text{ord}(x) \wedge \neg \Diamond_{\text{ord},<,H} \Big((\text{ord}', <') \leq (\text{ord}, <) \wedge \big(\exists u \mid H(u)\big)\big(\text{ord}'(u)\big) \wedge \mathcal{V}(V')\Big)]\Big) \qquad (K3)$$

Enter this $\Diamond_{\text{ord},<}$. By our assumption that ord, $<$ has a maximal element, it is enough to show that B is empty. For, if B is empty we can apply Simple Comprehension (*Axiom 8.4*) to define H to apply to the unique maximal element of ord, $<$ and then by *Lemma B.3 (Box Elimination)* we can infer the possibility of ord$'$, $<'$ equal to ord, $<$ such that $\mathcal{V}(V')$. We then use Simple Comprehension (*Axiom 8.4*) to $\Diamond_{V',\text{ord},<}$. define set, $\in$, @ to copy set$'$, $\in'$, @$'$ and then we may infer that $\mathcal{V}(V)$ and then infer the desired consequent by Diamond Collapsing (*Lemma B.4*).

So suppose, for contradiction, that B is non-empty. By the definition of well ordering *Definition E.2* there must be some least o in B. We first suppose that there is some maximal o^- satisfying $\text{ord}(o^-) \wedge \neg B(o^-)$ (or that o is the minimal element satisfying ord) and argue that we can extend the initial segment V^- of height o^- (i.e. o^- is the maximal element satisfying ord$^-$) guaranteed by the fact that $\neg B(o^-)$ into an initial segment V' of height o where $(\text{ord}', <') \leq (\text{ord}, <)$. This follows by the same reasoning used in the Proper Extension Lemma (*Lemma K.4*) to add a layer of classes to V^-. By straightforward, if tedious, application of Simple Comprehension (*Axiom 8.4*), *Lemma H.4 (Full Box Elimination)* and Diamond Collapsing (*Lemma B.8*) this contradicts the assumption that $B(o)$.

So suppose instead that there is no maximal o^- satisfying $\text{ord}(o^-) \wedge \neg B(o^-)$, i.e., o is a limit ordinal. We again, for contradiction, seek to construct a single initial segment V' of height o (i.e. o is the maximal element satisfying ord$'$) such that $(\text{ord}', <') \leq (\text{ord}, <)$. To this end we seek to derive the possibility of a single initial segment V_Σ such that $(<_\Sigma, \text{ord}_\Sigma) \leq (<, \text{ord})$ such that every $u < o$ satisfies $\text{ord}_\Sigma(u)$ from the logical possibility of segments V_u for $u < o$ witnessing the minimality of o. We may then apply the same reasoning above to extend V_Σ to V' by adding the ordinal o and applying the reasoning from the the Proper Extension Lemma (*Lemma K.4*).

We will do this by essentially the same argument used in the proof of *Proposition M.8 (Potentialist Replacement)* so we direct the reader to this proof to see the argument in greater detail.

Specifically, using *Axiom 8.4 (Simple Comprehension)* and *Axiom 8.9 (Modal Comprehension)* we let I apply to just those u less that o, i.e., those u such that $\neg B(u)$ (by the argument at the start of the proof $\neg B(u)$ must apply to some initial segment of ord). Now let Φ be the sentence expressing the claim that V' is an initial segment containing an ordinal satisfying Q and that $(\text{ord}', <') \leq (\text{ord}, <)$.

From equation *(K3)* we can straightforwardly derive the following sentence as it merely repackages the claim in equation *(K3)* that whenever H applies to some unique object failing to satisfy B then it's logically possible to have an initial segment V' containing an ordinal u with $(\text{ord}', <') \leq (\text{ord}, <)$.

$$\Box_{\text{ord},<,B,I}\,[\,(\exists!\,x \mid Q(x))(I(x)) \to \lozenge_{\text{ord},<,B,I,Q}\,\Phi]$$

Hence, we may invoke *Axiom 8.13 (Amalgamation)* to infer (where Υ is the sentence asserting that $V' = \hat{V}_x$ as in *Theorem K.1 (Hierarchy Combining)* .

$$\lozenge_{\text{ord},<,B,I}\,(\forall x)(\forall y)(\forall y')\big[\mathrm{y} \neq \mathrm{y}' \wedge\ \hat{V}(x,y) \wedge \hat{V}(x,y') \to x\epsilon\ Ext(W,<,B,I)\big] \wedge$$

$$\Box_{\text{ord},<,B,I,\hat{V}}\ [\,(\exists!\,x \mid Q(x))(I(x) \wedge \Upsilon(x)) \to \Phi]$$

After importing all necessary facts to the above context, we apply Diamond Collapsing (*Lemma B.4*) to collapse the contexts we've entered into a single $\lozenge_{\text{ord},<}$.

It is tedious, but relatively straightforward, to transform this result into the precondition for applying *Corollary K.4 (from the Hierarchy Combining Lemma)*. This lets us derive the logical possibility of a V_Σ and a function f such that every $\hat{V}_u$ is isomorphic (via f) to an initial segment of V_Σ where $(\text{ord}_\Sigma, <_\Sigma) \leq (\text{ord}, <)$.

We now argue that for each $u < o$ we have $\text{ord}_\Sigma(u)$. By the assumptions above (eliding the routine tasks of entering and leaving $\lozenge$ contexts) we know that f isomorphicly maps some $(\text{ord}', <') \leq (\text{ord}, <)$ such that $\text{ord}'(u)$ to $(\text{ord}_\Sigma, <_\Sigma) \leq (\text{ord}, <)$. By *Lemma K.1 (Well Ordering Agreement Lemma)* f must be the identity, vindicating the claim from the start of this paragraph. By the remark above, it now suffices to extend V_Σ by a single layer to V' with $\text{ord}'(o)$ to give the contradiction. ■

L. Translation Lemmas

A few key lemmas about interpreted initial segments will play a central role in all that proofs that follow.

L.1 Assignment Tweaking Lemma

First, note that potentialist translations tend to make claims about how arbitrary assignments V, ρ can be modified and extended, by some $V', \rho' \geq_u V, \rho$ changing only ρ's assignment of a single variable. *Lemma L.1 (Pointwise Tweaking)* lets us do this.

Lemma L.1 (Pointwise Interpretation Tweaking). *If $\Phi = (\exists x \mid \text{set}(x))(\phi(x))$ is a sentence without any $\lozenge$ or $\Box$ operators and is content restricted to $V, \rho, \mathcal{L}$ and u is a formal variable in our language of set theory and neither ρ' nor V' are in $\mathcal{L}$ then $\big(\vec{\mathcal{V}}(V,\rho) \wedge \Phi\big) \to \lozenge_{(V,\rho),\mathcal{L}}$ $\Big[(V,\rho') \underset{u}{\geq} (V,\rho) \wedge \phi(\rho'(\ulcorner u \urcorner)) \wedge \text{set}(\rho'(\ulcorner u \urcorner))\Big]$ Moreover, $\big(\vec{\mathcal{V}}(V,\rho) \wedge \Phi\big) \to \lozenge_{V,\rho,\mathcal{L}}$ $\Big[(V',\rho') \underset{u}{\geq} (V,\rho) \wedge \phi(\rho'(\ulcorner u \urcorner)) \wedge \text{set}(\rho'(\ulcorner u \urcorner)) \wedge V' = V\Big]$*

In the above lemma $V' = V$ is understood to abbreviate the claim that the relations $\in, <$ @ apply to exactly the same tuples as $\in', <'$@$'$.

Proof. We prove the moreover claim, as the primary claim trivially follows from it.

Suppose $\left(\vec{\mathcal{V}}(V,\rho) \wedge \Phi\right)$. By Simple Comprehension (*Axiom 8.4*) we can $\Diamond_{V,\rho,\mathcal{L}}$ define $P(x)$ to hold just if set$(x) \wedge \phi(x)$. Now by another application of Simple Comprehension and *Proposition 8.1 (Simple Choice)* we can ($\Diamond_{V,\rho,\mathcal{L},P}$) have Q select a unique object such that $P(x)$. By multiple applications of Simple Comprehension (*Axiom 8.4*) we can ($\Diamond_{V,\rho,\mathcal{L},P,Q}$) have $V' = V$ and $\rho' = \rho$ excepting only that $\rho'(\ulcorner u \urcorner)$ is chosen to be the unique object satisfying Q and thus $\phi(\rho'(\ulcorner u \urcorner))$. By using *Lemma H.7 (Multiple Definitions)* to coordinate the above applications of *Axiom 8.4 (Simple Comprehension)* and *Proposition 8.1 (Simple Choice)* we can infer the desired conclusion

$$\Diamond_{V,\rho,\mathcal{L}} \left[(V',\rho') \underset{u}{\geq} (V,\rho) \wedge \phi(\rho'(\ulcorner u \urcorner)) \wedge \text{set}(\rho'(\ulcorner u \urcorner)) \wedge V' = V\right]$$

■

We can also prove the following lemma about how it's possible ($\Diamond_{\mathcal{L},V}$) to transform any non-empty initial segment V into an interpreted initial segment $\vec{V}$.

Lemma L.2 (Interpretation Adding). *If $\Phi = (\exists x \mid \text{set}(x))(\phi(x))$ is a sentence content restricted to $V', \mathcal{L}$ not containing any $\Diamond$ or $\Box$ operators and u is a formal variable in our language of set theory and neither $\rho, \mathbb{N}, S$ are in $V, \mathcal{L}$. Then*

$$(\mathcal{V}(V) \wedge \Phi) \rightarrow \Diamond_{V,\mathcal{L}}\left[\vec{\mathcal{V}}(V,\rho) \wedge \phi(\rho(\ulcorner u \urcorner))\right]$$

Proof. Trivially $(\exists x)(\text{set}(x))$ so by *Corollary K.2 (Interpreted Extension)* we can infer $\Diamond_{V,\mathcal{L}} \vec{\mathcal{V}}(\vec{V})$. Enter this context and apply *Lemma L.1 (Pointwise Tweaking)* to derive that

$$\Diamond_{V,\rho,\mathcal{L}} \quad \left[(V',\rho') \underset{u}{\geq} (V,\rho) \wedge \phi(\rho'(\ulcorner u \urcorner)) \wedge \text{set}(\rho'(\ulcorner u \urcorner)) \wedge V' = V\right]$$

Entering this $\Diamond_{V,\rho,\mathcal{L}}$ context. It is trivial to infer that

$$\vec{\mathcal{V}}(V,\rho') \wedge \phi(\rho'(\ulcorner u \urcorner))$$

Leaving all $\Diamond$ contexts and applying Diamond Collapsing (*Lemma B.4*) gives us

$$\Diamond_{V,\mathcal{L}} \mathcal{V}\,(V,\rho') \wedge \phi(\rho'(\ulcorner u \urcorner))$$

The desired conclusion follows easily by Relabeling (*Axiom 8.5*). ■

L.2 Translation Theorem

Next, we can prove a translation theorem which says that the way V_n, ρ_n assigns the free variables in a set theoretic formula θ completely determine the truth value of $t_n(\theta)$ (i.e., the (partial) potentialist translation of the formula θ)[24].

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[24] Note that Hellman proves something analogous to this lemma in (Geoffrey 1996), assuming the axiom of inaccessibles (but I make no such assumption)

First, however, we establish a few useful utility lemmas about translations.

Lemma L.3 (Renumbering). *Suppose that $n+k$ is the maximal integer such that (any part of) $\vec{V}_{n+k}$ is mentioned in $t_n(\theta)$, i.e., θ has quantifiers nested to depth k, then $t_n(\theta)[\vec{V}_n/\vec{V}_m, \ldots, \vec{V}_{n+k}/\vec{V}_{m+k}]$*

Proof. This follows via a straightforward structural induction on θ using the . ■

Lemma L.4 (Coextensive Hierarchies Lemma). *If $\vec{V}_n = \vec{V}_m$ then $(t_n(\theta) \leftrightarrow t_m(\theta))$.*

Proof. It is enough to prove the → direction as the other direction follows by swapping the values of n and m. So suppose $\vec{V}_n = \vec{V}_m$ and $t_n(\theta)$ we argue, by structural induction on θ, that $t_m(\theta)$ holds. So suppose the claim holds for all n, m on all subformulas of θ.

The only interesting case is when $\theta = (\exists x)\phi(x)$ since the atomic case is trivial and t_n, t_m commute with truth-functional operations. In this case

$$t_n(\theta) = \Diamond_{\vec{V}_n} \left(\vec{V}_{n+1} \underset{x}{\geq} \vec{V}_n \wedge t_{n+1}(\phi)\right)$$

Enter this $\Diamond_{\vec{V}_n}$ context and apply Simple Comprehension (*Axiom 8.4*) (via the Multiple Definitions Lemma (*Lemma H.7*)) to define $\vec{V}_{m+1} = \vec{V}_{n+1}$ giving us

$$\Diamond_{\vec{V}_n} \Diamond_{\vec{V}_n, \vec{V}_{n+1}} \vec{V}_{n+1} \underset{x}{\geq} \vec{V}_n \wedge \vec{V}_{n+1} = \vec{V}_{m+1} \wedge t_{n+1}(\phi)$$

Using the inductive hypothesis inside the $\Diamond_{\vec{V}_n} \Diamond_{\vec{V}_n, \vec{V}_{n+1}}$ context and then applying Diamond Collapsing (*Lemma B.8*) lets us infer

$$\Diamond_{\vec{V}_n} \vec{V}_{m+1} \underset{x}{\geq} \vec{V}_n \wedge t_{m+1}(\phi) \qquad (L1)$$

To finish the proof we need only replace the $\vec{V}_n$ in the above equation with $\vec{V}_m$. Intuitively, this follows immediately from the assumption that $\vec{V}_m = \vec{V}_n$ but we verify this formally using the *Theorem I.1 (Isomorphism Lemma)*.

By Simple Comprehension (*Axiom 8.4*) via the Multiple Definitions Lemma (*Lemma H.7*) we may define Z be the identity relation on $\vec{V}_n$ while pulling in equation *(L1)* and the fact that $\vec{V}_n = \vec{V}_m$. Enter this $\Diamond_{\vec{V}_n}$ context. Since $\vec{V}_n = \vec{V}_m$ we have $\vec{V}_n \cong_Z \vec{V}_m$. As equation *(L1)* is content restricted to $\vec{V}_n$ we may thus apply the Isomorphism Theorem (*Theorem I.1*) to replace $\vec{V}_n$ with $\vec{V}_m$ in *(L1)* letting us infer

$$t_m(\theta) = \Diamond_{\vec{V}_m} \vec{V}_{m+1} \underset{x}{\geq} \vec{V}_n \wedge t_{m+1}(\phi)$$

The desired conclusion follows by applying *Axiom 8.2 (Diamond Elimination)* to export $t_m(\theta)$ from the $\Diamond_{\vec{V}_n}$ context. ■

Theorem L.1 (Translation Theorem). *If $v_1 \ldots v_k$ are the only variables free in a set theoretic formula θ, then* $\rightarrow (t_n(\theta) \leftrightarrow t_m(\theta))$

We note that the theorem above, as one would expect[25], is indifferent to the particular relations used for V_n and V_m. So, for instance, the following conclusion also holds (where $(t_n^*(\theta) \stackrel{\text{def}}{=} t_n(\theta)[\vec{V}_n/\vec{V}^*]$ and $t'_m(\theta) \stackrel{\text{def}}{=} t_m(\theta)[\vec{V}_m/\vec{V}']$)

$$\rightarrow (t_n^*(\theta) \leftrightarrow t'_m(\theta))$$

Proof. I will prove this claim by induction on formula complexity. So suppose the sentence specified above is provable for all subformulas of θ and choices of $V_n, V_m, \rho_n, \rho_m v_1, \ldots v_k$, as in the statement of the lemma.

When θ is an atomic sentence, i.e., one of the form $x = y$ or $x \in y$, the claim clearly holds for θ since $t_n(x = y)$ is $\rho_n(\ulcorner x \urcorner) = \rho_n(\ulcorner y \urcorner)$ and $t_n(x \in y)$ is $\rho_n(\ulcorner x \urcorner) \in_n \rho_n(\ulcorner y \urcorner)$. Also when θ is a truth-functional combination of other formulas then the claim holds for θ since t_n and t_m commute with truth-functional operators.

The only non-trivial case is when $\theta = (\exists x)\phi(x)$ (as we take $\forall x$ to abbreviate $\neg\exists x\neg$). Note that it is enough to show the $\rightarrow$ direction as the other direction follows by switching the values of n and m. Our strategy will be to use the *Lemma K.7 (Hierarchy Extending)* to replace $\vec{V}_{n+1}$ in $t_n(\theta)$ with a $\vec{V}_{m+1} \geq_x \vec{V}_m$ extending some $\vec{V}' \cong \vec{V}_{n+1}$ (so ρ_{m+1} agrees with ρ'). We will then use the *Theorem I.1 (Isomorphism Lemma)* to infer $t'_{n+1}(\phi)$ where $t'_{n+1}(\phi) \stackrel{\text{def}}{=} t_{n+1}(\phi)[\vec{V}_{n+1}/\vec{V}']$ and then use the inductive hypothesis to infer $t_{m+1}(\phi)$.

So suppose that

$$t_n(\theta) = \Diamond_{\vec{V}_n} \vec{V}_{n+1} \underset{x}{\geq} \vec{V}_n \wedge t_{n+1}(\phi)$$

.

By $\Diamond$ Ignoring (*Axiom 8.3*) we can infer

$$\Diamond_{\vec{V}_n,\vec{V}_m} \vec{V}_{n+1} \underset{x}{\geq} \vec{V}_n \wedge t_{n+1}(\phi)$$

Enter this $\Diamond_{\vec{V}_n,\vec{V}_m}$ context and import the assumptions of the theorem, e.g., $V_n \geq V_m \vee V_m \geq V_n$ .... We now seek to define $\vec{V}_{m+1}$. Note that without loss of generality we may assume that

---

[25] While we always take the particular relation names mentioned in theorems or lemmas to be placeholders which can be instantiated with whatever relation names we wish we make specific mention of it here, due to the confusing interaction of the subscripts (which are part of the meta-lanaguage and *not* pure placeholders) and the relation names. So note that the results of any theorem still hold if we replace V_4 with V'_4 but we have no such guarantee if we replace V_4 with V_3, since the translations make explicit reference to particular numerical values of these subscripts.

$V_m \geq V_n$ since if $V_n \geq V_m$ we could simply set $\vec{V}_{m+1}$ to be equal to $\vec{V}_{n+1}$ and directly apply the inductive step.

Thus, assuming that $V_m \geq V_n$, we apply *Lemma K.7 (Hierarchy Extending)* to establish the possibility of a $V_{m+1} \geq V_m$. Letting $\mathcal{L}$ be $\{\vec{V}_n, \vec{V}_{n+1}, \vec{V}_m\}$ and taking the advantage of the moreover claim from *Lemma K.7 (Hierarchy Extending)* to ensure disjointness of V_{m+1} we have

$$\begin{aligned} \Diamond_{\mathcal{L}} V_{m+1} \quad & \geq V_m \wedge V_{m+1} \geq V' \wedge V_{n+1} \underset{f}{\cong} V' \wedge \\ & (\forall x \mid \mathrm{Ext}(\mathcal{L})(x))(V_{m+1}(x) \to V_{n+1}(x)) \end{aligned}$$

Import all necessary facts into this $\Diamond_{\mathcal{L}}$ context and then, exiting both the $\Diamond_{\mathcal{L}}$ context and the $\Diamond_{\vec{V}_n, \vec{V}_m}$ context we may apply Diamond Collapsing (*Lemma B.8*) and then reenter the single

$\Diamond_{\vec{V}_n, \vec{V}_m}$ context.

Inside the $\Diamond_{\vec{V}_n, \vec{V}_m}$ context we now construct ρ', ρ_{m+1} so that $\vec{V}' \cong_f \vec{V}_{n+1}$ and $\vec{V}' \leq_x \vec{V}_{m+1}$. To this end we use the Multiple Definitions Lemma (*Lemma H.7*) to pack together various applications of Simple Comprehension (*Axiom 8.4*), we can ($\Diamond_{\mathcal{L}, V_{m+1}}$) define ρ' to be the image of ρ_{n+1} under f. Finally, we let ρ_{m+1} agree with ρ' on 'x' but ρ_m everywhere else[26].

Since $V' \cong_f V_{n+1}$ and ρ' was defined to be the isomorphic image of ρ_{n+1} we have $\vec{V}' \cong_f \vec{V}_{n+1}$. Thus, using the *Theorem H.1 (Isomorphism Lemma)* we can infer $t_{n+1}'(\theta) \stackrel{\text{def}}{=} t_{n+1}(\phi)[\vec{V}_{n+1}/\vec{V}']$ from $t_{n+1}(\phi)$.

We now argue that $\rho' = \rho_{m+1}$ on all variables free in ϕ. By construction $\rho_{m+1}(\ulcorner x \urcorner) = \rho'(\ulcorner x \urcorner)$ and any other variable v free in ϕ must also be free in θ. Hence, $\rho_n(\ulcorner v \urcorner) = \rho_m(\ulcorner v \urcorner) = \rho_{m+1}(\ulcorner v \urcorner)$. As $V_n \leq V_{n+1}$ and, by assumption, $V_n \leq V_m \leq V_{m+1}$ we have $V_n \leq V_{n+1}, V_{m+1}$. Hence, as $V_{n+1} \cong_f V' \leq V_{m+1}$, by the *Lemma K.2 (Hierarchy Agreement),* f must be the identity on V_n and thus $\rho'(\ulcorner v \urcorner) = f(\rho_{n+1}(\ulcorner v \urcorner)) = \rho_{n+1}(\ulcorner v \urcorner) = \rho_n(\ulcorner v \urcorner) = \rho_{m+1}(\ulcorner v \urcorner)$.

Thus, by the inductive hypothesis we can infer $t_{m+1}(\phi)$ from $t'_{n+1}(\phi)$. As ρ_{m+1} was defined to agree with ρ_m on all variables but x we have

$$\vec{V}_{m+1} \underset{x}{\geq} \overrightarrow{V_m} \wedge t_{m+1}(\phi)$$

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More formally, that is

$$(\forall z \mid \mathbb{N}(z))\rho_{m+1}'(z) = \begin{cases} \rho'(\ulcorner x \urcorner) & \text{if } z = \ulcorner x \urcorner \\ \rho_m(z) & \text{otherwise} \end{cases}$$

Now leaving the above $\lozenge_{\vec{V}_n,\vec{V}_m}$ context and applying Reducing (*Lemma B.4*) to drop the $\vec{V}_n$ subscript yields our desired conclusion

$$t_m(\theta) = \lozenge_{V_m} V_{m+1} \underset{x}{\geq} V_m \wedge t_{m+1}(\phi)$$

■

We now argue that we can generalize the above result by weakening the assumption that V_n and V_m are compatible.

Corollary L.1 (Generalized Translation Lemma). *If $v_1 \ldots v_k$ are the only variables free in a set theoretic formula θ and V_0 is a non-empty initial segment then*

$$\begin{aligned} V_n \geq V_0 \quad & \wedge V_m \geq V_0 \wedge \vec{\mathcal{V}}(V_n) \wedge \vec{\mathcal{V}}(V_m) \ \wedge \mathrm{set}_0(\rho_n(\ulcorner v_1 \urcorner)) \wedge \ldots \wedge \mathrm{set}_0(\rho_n(\ulcorner v_k \urcorner)) \\ & \wedge \rho_n(\ulcorner v_1 \urcorner) = \rho_m(\ulcorner v_1 \urcorner) \wedge \ldots \rho_n(\ulcorner v_k \urcorner) = \rho_m(\ulcorner v_k \urcorner) \rightarrow (t_n(\theta) \leftrightarrow t_m(\theta)) \end{aligned}$$

Note that if $\vec{V}_0$ is an interpreted initial segment then V_0 is automatically non-empty. Also, the same remarks about t'_n and t^*_m from the *Theorem L.1 (Translation Theorem)* apply here

Proof. Again it is enough to show that $t_n(\theta) \rightarrow t_m(\theta)$ under the conditions above since the other direction follows by switching n and m. So suppose that the conditions of the corollary are met, the antecedent holds and $t_n(\theta)$ holds.

By Simple Comprehension (*Axiom 8.4*) we can ($\lozenge_{\vec{V}_n,\vec{V}_m}$) define ρ_0 to agree with ρ_n on all variables free in θ and to be the emptyset on all other values. Clearly $\vec{V}_0$ is an interpreted initial segment. Entering this $\Diamond_{\vec{V}_n,\vec{V}_m}$ context we now apply the *Theorem L.1(Translation)* twice to infer from $t_n(\theta)$ to $t_0(\theta)$ to $t_m(\theta)$. As $t_m(\theta)$ is content restricted to $\vec{V}_m$ we may use *Axiom 8.2 (Diamond Elimination)* to infer it holds outside the $\lozenge_{\vec{V}_n,\vec{V}_m}$ context as desired. ■

We also prove that changing the variables in a sentence to be translated doesn't change the truth-value of the translation provided the assignment function assigns the same value to both variables.

Lemma L.5 (Variable Swap Lemma). *If $\vec{V}_i$ is an interpreted initial segment with $\rho_i(v) = \rho_i(v')$, ϕ a set theoretic formula and ϕ' is the result of replacing zero or more occurrences of v in ϕ with v', provided that no bound variables are replaced, and all substituted occurrences of v' are free then $t_i(\phi) \leftrightarrow t_i(\phi')$*

Proof. We argue by induction on formula complexity. Suppose the assumptions in the lemma holds and the claim is provable for all subformula of ϕ. The claim is trivial if ϕ is atomic as well as if ϕ is a truthfunctional combination of subformula.

Suppose ϕ is $(\exists x)\psi(x)$. If either v or v' is x, we have $\phi = \phi'$ (and the desired result is immediate). For if $v = x$ then there are no free instances of x in $(\exists x)\psi(x)$ to replace, and if $v' = x$ then replacing any variable v in ψ with x in $(\exists x)\psi(x)$ result in capture.

So it remains to consider the case where v or v' are distinct variables from x. Assume that $t_i((\exists x)\psi(x))$ holds. By the definition of t_n (*Definition 6.1 Potentialist Translation*) we have

$$\Diamond_{\vec{V}_i}\left[\vec{V}_{i+1} \underset{x}{\geq} \vec{V}_i \wedge t_{i+1}(\psi(x))\right]$$

Enter this $\Diamond_{\vec{V}_n}$ context. Because v, v' are distinct from x, we can infer $\phi' = [(\exists x)\psi'(x)]$ for some ψ' where ψ' replaces some instances of v (which are free in ψ because they are free in $(\exists x)\psi$) with instances of v' (which are free in ψ' because they are free in $(\exists x)\psi'$). And because v, v' are distinct from x we have $\rho_{i+1}(v) = \rho_{i+1}(v')$. Thus, by the inductive hypothesis we can infer $t_{i+1}(\psi'(x))$. Exiting the $\Diamond_{\vec{V}_n}$ context yields, by the definition of t_n (*Definition 6.1 Potentialist Translation*) $t_i((\exists x)\psi'(x)) = t_i(\phi')$. The same argument lets us derive $t_i(\phi)$ on the assumption that $t_i(\phi')$ completing the proof. ■

L.3 Bounded Quantifiers Lemma

Definition L.1. *Working in the language of set theory we, say that a quantifier is bounded if it has the form $(Qx \in y)$ where*

$$\begin{aligned}(\forall x \in y)\phi &\overset{\text{def}}{\leftrightarrow} \quad (\forall x)(x \in y \rightarrow \phi) \leftrightarrow \neg(\exists x \in y)\neg\phi \\ (\exists x \in y)\phi &\overset{\text{def}}{\leftrightarrow} \quad (\exists x)(x \in y \wedge \phi)\end{aligned}$$

And we say that a formula ϕ in the language of set theory is bounded if all quantifiers appear in ϕ are bounded.

Note that all bounded quantifiers can be written in terms of bounded existential quantification, e.g., $(\forall x \in y)\phi \leftrightarrow \neg(\exists x \in y)\neg\phi$ so we may assume that all bounded quantifiers are existential .

We now argue that bounded formulas can be translated in a particularly simple way.

Definition L.2 (Bounded Translation). *If ϕ is a formula of set theory and V an initial segment we define ϕ^V to be the result of replacing all occurrences of $\in$ in ϕ with $\in_n$. Furthermore, if $\vec{V}$ is an interpreted initial segment and $x_1, \dots x_n$ are the variables free in ϕ we define $\phi^{\vec{V}} = \phi^V(\rho(\ulcorner x_0 \urcorner), \dots, \rho(\ulcorner x_n \urcorner))$*

Recall the following lemma from Appendix *A.*

Lemma L.6 (Bounded Quantifiers Lemma). *Suppose ϕ is a bounded formula in the language of set theory and $\vec{V}_n$ is an interpreted initial segment (as per* Definition A.4 *) then $\phi^{\vec{V}_n} \leftrightarrow t_n(\phi)$*

Proof. Assume that $x_0, \dots x_n$ are the only variables free in ϕ we need to show

$$\phi^{V_n}(\rho(\ulcorner x_0 \urcorner), \dots, \rho(\ulcorner x_n \urcorner)) \leftrightarrow t_n(\phi)$$

Note that we may assume that no variables appear both free and bound in ϕ since, by renaming bound variables, any first order logical formula is first order provably equivalent to one with this property and, by *Theorem L.1 (Translation Theorem)*, this equivalence carries over to t_n translations.

We now prove the lemma via induction on formula complexity. If ϕ is quantifier-free then by the definition of t_n (*Definition 1.2*) it is apparent that

$$\phi^{V_n}(\rho_n(\ulcorner x_1 \urcorner), \dots, \rho_n(\ulcorner x_n \urcorner)) \leftrightarrow \phi(\rho_n(\ulcorner x_1 \urcorner), \dots, \rho_n(\ulcorner x_n \urcorner))[\in/\underset{n}{\in}] \leftrightarrow t_n(\phi)$$

Now suppose the lemma holds for all subformulas of ϕ we establish that it holds for ϕ as well. The only difficult case is where $\phi = (\exists y \in x_i)\psi$.

In this case, note that $\phi^{V_n}(\rho_n(\ulcorner x_0 \urcorner), \dots, \rho_n(\ulcorner x_n \urcorner))$ is equivalent to

$$\left(\exists y \underset{n}{\in} \rho_n(\ulcorner x_i \urcorner)\right) \psi^{V_n}(\rho_n(\ulcorner x_0 \urcorner), \dots, \rho_n(\ulcorner x_n \urcorner, y)) \qquad (L2)$$

.

Assuming that the above formula holds we may apply *Lemma L.1 (Pointwise Tweaking)* and define $\vec{V}_{n+1}$ so that $\rho_{n+1}(\ulcorner y \urcorner) \in_n \rho_n(\ulcorner x_i \urcorner)$, i.e., we can deduce that

$$\begin{aligned} \Diamond_{\vec{V}_n} [\quad & \vec{V}_{n+1} \underset{y}{\geq} \vec{V}_n \wedge \rho_{n+1}(\ulcorner y \urcorner) \underset{n}{\in} \rho_n(\ulcorner x_i \urcorner) \wedge \\ & \psi^{V_n}(\rho_n(\ulcorner x_0 \urcorner), \dots, \rho_n(\ulcorner x_n \urcorner), \rho_{n+1}(\ulcorner y \urcorner))] \qquad (L3) \end{aligned}$$

Note that this formula is actually equivalent to *(L2)* since (working inside the $\Diamond_{\vec{V}_n}$ context) we may derive *(L2)* and then use *Axiom 8.2 (Diamond Elimination)* to export this conclusion.

Enter the above $\Diamond_{\vec{V}_n}$ context. As $\vec{V}_{n+1} \geq_y \vec{V}_n$ using the definition of we can infer that equation *(L3)* is equivalent to

$$\begin{aligned} \vec{V}_{n+1} \underset{y}{\geq} \vec{V}_n \quad & \wedge \rho_{n+1}(\ulcorner y \urcorner) \underset{n+1}{\in} \rho_{n+1}(\ulcorner x_i \urcorner) \wedge \\ & \psi^{V_{n+1}}(\rho_{n+1}(\ulcorner x_0 \urcorner), \dots, \rho_{n+1}(\ulcorner x_n \urcorner), \rho_{n+1}(\ulcorner y \urcorner)) \end{aligned}$$

By the inductive assumption applied to ψ this is equivalent to

$$\vec{V}_{n+1} \underset{y}{\geq} \vec{V}_n \wedge \rho_{n+1}(\ulcorner y \urcorner) \in \rho_{n+1}(\ulcorner x_i \urcorner) \wedge t_{n+1}(\psi)$$

By the this is equivalent to

$$\vec{V}_{n+1} \underset{y}{\geq} \vec{V}_n \wedge t_{n+1}(y \in x_i \wedge \psi)$$

So leaving the $\Diamond_{V,\rho}$ context we have

$$\Diamond_{\overrightarrow{V_n}} [\overrightarrow{V_{n+1}} \underset{y}{\geq} \overrightarrow{V_n} \wedge t_{n+1}(y \in x_i \wedge \psi)] \overset{\text{def}}{\leftrightarrow} t_n(\phi)$$

Since all the steps were equivalences, this suffices to prove the lemma. ■

L.4 Translation Simplification Lemmas

The following lemma shows that our official potentialst paraphrases turn out to be logically equivalent to simpler (and more traditional) potentialist paraphrases as below.

Lemma L.7 (Existential Potentialist Translation). $t((\exists x)\phi) \leftrightarrow \Diamond \left[\vec{\mathcal{V}}(\vec{V}_1) \wedge t_1(\phi)\right]$

Proof. (→) Suppose that $t((\exists x)\phi)$ i.e.,

$$\left[\vec{\mathcal{V}}(\vec{V}_0) \rightarrow \Diamond_{\vec{V}_0}\left(\vec{V}_1 \geq \vec{V}_0 \wedge t_1(\phi)\right)\right] \qquad (L4)$$

By Interpreted Initial Segment Possibility (*Lemma K.5*), we have $\Diamond\, \vec{\mathcal{V}}(\vec{V}_0)$ Entering this $\Diamond$ context, by *Axiom 8.6 (Importing)* we may import *(L4)* as it is content restricted to the empty list. The desired conclusion now follows straightforwardly by application of *Lemma B.3 (Box Elimination)*, modus ponus and Diamond Collapsing (*Lemma B.8*).

←. Suppose $\Diamond\left[\vec{\mathcal{V}}(\vec{V}_1) \wedge t_1(\phi(x))\right]$. Note that this assumption is content restricted to the empty list and thus can be assumed for purposes of *Lemma B.2 (Box Introduction)*. We now seek to prove. $\Diamond_{\vec{V}_n}\left(\vec{V}_1 \geq \vec{V}_0 \wedge t_1(\phi)\right)$ from the assumption $\vec{\mathcal{V}}(\vec{V}_0)$.

By Relabeling (*Axiom 8.5*) and $\Diamond$ Ignoring (*Axiom 8.3*) we can infer (remember $t_1^*(\phi) \overset{\text{def}}{\leftrightarrow} t_1(\phi)[\vec{V}_1/\vec{V}^*]$)

$$\Diamond_{\vec{V}_0}\left(\vec{\mathcal{V}}(\vec{V}^*) \wedge t_1^*(\phi)\right) \qquad (L5)$$

Enter this $\Diamond_{\vec{V}_0}$ and import the fact that $\vec{\mathcal{V}}(\vec{V}_0)$. By *Lemma K.7 (Hierarchy Extending)*, we can infer the possibility $\Diamond_{V_0,V^*}$ of an initial segment V_1 which extends both V_0 and V', where V' is the isomorphic image of V^* (under f). Using $\Diamond$ Ignoring (*Axiom 8.3*) we may add the subscript ρ_0, ρ^* to this possibility claim giving us

$$\Diamond_{V_0,V^*,\rho_0,\rho^*} V_1 \geq V_0 \wedge V_1 \geq V' \wedge V' \underset{f}{\cong} V^*$$

Enter this $\Diamond_{\vec{V}_0,\vec{V}^*,\rho_0,\rho^*}$ context and import the interior of *(L5)* . Via the Multiple Definitions Lemma (*Lemma H.7*) and Simple Comprehension (*Axiom 8.4*), we can further derive the possibility ($\Diamond_{\vec{V}_0,\vec{V}^*,f,V'}$) that the facts derived in this context remain true while defining ρ' to be the image of ρ^* under f and ρ_1 is defined to match ρ_0 everywhere but on $\ulcorner x \urcorner$ where we set $\rho_1(\ulcorner x \urcorner) = \rho'(\ulcorner x \urcorner)$. This entails both that $\vec{V}' \cong_f \vec{V}^*$ and $\vec{V}_1 \geq_x \vec{V}_0$.

We may now apply the *Theorem I.1 (Isomorphism Lemma)* to infer $t'_1(\phi)$ from $t_1^*(\phi)$ and the *Theorem L.1 (Translation Lemma)* to infer $t_1(\phi)$. Leaving all the $\Diamond$ contexts we are inside and applying Diamond Collapsing (*Lemma B.8*) we infer.

$$\Diamond_{\vec{V}_0} \vec{V}_1 \geq \vec{V}_0 \wedge t_1(\phi)]$$

Canceling the assumption of $\vec{\mathcal{V}}(\vec{V}_0))$ and applying *Lemma B.2 (Box Introduction)* gives us our desired conclusion of

$$\Box[\vec{\mathcal{V}}(\vec{V}_0) \rightarrow \Diamond_{\vec{V}_0} \vec{V}_1 \geq \vec{V}_0 \wedge t_1(\phi)]$$

■

We can establish equivalence between the longer and shorter translations for universal claims $\forall x \phi(x)$.

Lemma L.8 (Universal Potentialist Translation). $t((\forall x)\phi) \leftrightarrow \Box(\vec{\mathcal{V}}(\vec{V}_1) \to t_1(\phi(x))$

Proof. Note that by expanding out the left hand side and applying *Lemma H.5 (Box Simplification)* it is enough to show

$$\Box\left[\vec{\mathcal{V}}(\vec{V}_0) \wedge \vec{V}_1 \underset{x}{\geq} \vec{V}_0 \to t_1(\phi)\right] \leftrightarrow \Box[\vec{\mathcal{V}}(\vec{V}_1) \to t_1(\phi(x)]$$

($\leftarrow$) This follows trivially by *Lemma H.2 (Box Closure)*.

($\rightarrow$) We prove the contrapositive. So suppose that

$$\neg\Box[\vec{\mathcal{V}}(\vec{V}_1) \to t_1(\phi(x)]$$

Or, equivalently,

$$\Diamond\left[\vec{\mathcal{V}}(\vec{V}_1) \wedge \neg t_1(\phi(x)\right]$$

Enter this $\Diamond$ context. By the the Multiple Definitions Lemma (*Lemma H.7*) and Simple Comprehension (*Axiom 8.4*) it is possible ($\Diamond_{\vec{V}_0}$) to 'define' $\vec{V}_0 = \vec{V}_1$. Entering this $\Diamond_{\vec{V}_0}$ context we may immediately infer

$$\vec{\mathcal{V}}(\vec{V}_1) \wedge \vec{V}_1 \underset{x}{\geq} \vec{V}_0 \wedge \neg t_1(\phi(x)$$

Leaving both $\Diamond$ contexts we have entered and applying Diamond Collapsing (*Lemma B.8*) lets us infer

$$\Diamond\, \vec{\mathcal{V}}(\vec{V}_1) \wedge \vec{V}_1 \underset{x}{\geq} \vec{V}_0 \wedge \neg t_1(\phi(x)$$

Or, equivalently,

$$\neg\Box[\vec{\mathcal{V}}(\vec{V}_0) \wedge \vec{V}_1 \underset{x}{\geq} \vec{V}_0$$

as desired. ■

M Detailed Justification of ZFC Axioms

M.1 Translation Equivalence Lemma

Lemma M.1 (Translation Equivalence Lemma). *If $\Phi = (\forall v_1) \ldots (\forall v_n)\phi$ is a sentence in the language of set theory then* $\Box\left[\vec{\mathcal{V}}(\vec{V}_n) \to t_n(\phi)\right] \to t(\Phi)$

Proof. We prove this claim by induction on n. If $n = 0$ then Φ is just ϕ and the antecedent is

$$\Box\left[\vec{\mathcal{V}}(\vec{V}_0) \to t_0(\phi)\right]$$

which is just $t(\phi) = t(\Phi)$.

We now suppose the claim holds for n and prove that it holds for $n+1$. So suppose that

$$\Box[\vec{\mathcal{V}}(\vec{V}_{n+1}) \rightarrow t_{n+1}(\phi)]$$

Since $\vec{V}_{n+1} \geq_{v_{n+1}} \vec{V}_n$ implies $\vec{\mathcal{V}}(\vec{V}_{n+1})$ by *Lemma H.2 (Box Closure)* we may infer

$$\Box[\vec{\mathcal{V}}(\vec{V}_n) \wedge \vec{V}_{n+1} \underset{v_{n+1}}{\geq} \vec{V}_n \rightarrow t_{n+1}(\phi)]$$

As $\vec{\mathcal{V}}(\vec{V}_n)$ is content restricted to $\vec{V}_n$ and by *Lemma H.6 (Diamond Simplification)* (the equivalent formulation in the theorem) to infer

$$\Box[\vec{\mathcal{V}}(\vec{V}_n) \rightarrow \Box_{\vec{V}_n}\left[\vec{V}_{n+1} \underset{v_{n+1}}{\geq} \vec{V}_n \rightarrow t_{n+1}(\phi)\right]$$

However, by this is just

$$\Box[\vec{\mathcal{V}}(\vec{V}_n) \rightarrow t_n((\forall v_{n+1})\phi)]$$

The desired conclusion now follows by the inductive hypothesis. ■

M.2 Foundation

Before we prove potentialist foundation we establish the following lemma.

Lemma M.1. *If V is an initial segment and x is a non-empty set in V then there is some $y \in x$ such that* $(\forall z \mid z \in x)(z \notin y)$.

Proof. Suppose, for contradiction, V is an initial segment and the claim fails for some set x in V. By the the Multiple Definitions Lemma (*Lemma H.7*), Simple Comprehension (*Axiom 8.4*) and *Proposition 8.1 (Simplified Choice)* we can ($\lozenge_V$) have Q_x apply to a single x witnessing this failure as well as (by another application of Simple Comprehension (*Axiom 8.4*)) have $B(o)$ apply to exactly those ordinals o such that $(\exists x \mid Q_x(x))(\exists y \in x)@(y, o)$. Enter this $\lozenge_V$ context.

Now by the fact that ord, $<$ is a well ordering by the and *Lemma B.3*, it follows that there exists a $<$ least element satisfying B. Let o be this object. Then o is the $<$ least object in ord such that $(\exists y \in x)@(y, o)$. Let y be a witness to this existential. We claim that y satisfies the lemma, i.e., witnesses the contradiction with the assumption the lemma fails.

Consider any $z \in y$. By , if $z \in y$ and $@(y, o)$ then there is some $o' < o$ with $@(z, o')$. So if z were in x this would contradict the minimality of o. Thus, we may exit the $\lozenge_V$ context entered above and using *Axiom 8.2 (Diamond Elimination),* export the contradiction to establish the lemma. ■

We now prove *Proposition M.2 (Potentialist Foundation)*.

Proposition M.2 (Potentialist Foundation). $t((\forall x)[(\exists a)(a \in x) \to (\exists y)(y \in x \wedge \neg(\exists z)(z \in y \wedge z \in x))])$

Proof. By *Lemma M.1 (Translation Equivalence Lemma)* it suffices to prove the following:

$$\Box[\vec{\mathcal{V}}(\overrightarrow{V_1}) \to t_1(\phi)] \text{ where}$$
$$\phi = (\exists a \in x) \to (\exists y \in x)(\forall z \in x)\neg(z \in y)$$

Suppose that $\vec{\mathcal{V}}(\overrightarrow{V_1})$. The formula ϕ is bounded. So by the *Lemma L.6 (Bounded Quantifiers)* , to prove $t_1(\phi)$ it is enough to prove

$$\phi^{\vec{V}_1} \overset{\text{def}}{\leftrightarrow} \left(\exists a \underset{1}{\in} \rho_1(\ulcorner x \urcorner)\right) \to \left(\exists y \underset{1}{\in} \rho_1(\ulcorner x \urcorner)\right)(\forall z \underset{1}{\in} \rho_1(\ulcorner x \urcorner))\neg(z \underset{1}{\in} y)$$

Assume $\exists a \in_1 \rho_1(\ulcorner x \urcorner)$. As $\rho_1(\ulcorner x \urcorner))$ is a set in V_1 and non-empty by *Lemma M.1 (Translation Equivalence Lemma)* $\phi^{V_1}(\rho_1(\ulcorner x \urcorner))$ holds, establishing $\phi^{\vec{V}_1}$.

Thus we've shown

$$\vec{\mathcal{V}}(\overrightarrow{V_1}) \to t_1(\phi)$$

The result now follows by *Lemma B.2 (Box Introduction)* ∎

M.3 Extensionality

Proposition M.2 (Extensionality). $t((\forall x)(\forall y)([(\forall z \in x)(z \in y) \wedge (\forall z \in y)(z \in x)] \to x = y))$

Proof. By the *Lemma M.1 (Translation Equivalence Lemma)* it suffices to prove the following:

$$\Box[\vec{\mathcal{V}}(\vec{V}_2) \to t_2([(\forall z \in x)(z \in y) \wedge (\forall z \in y)(z \in x)] \to x = y)]$$

So consider, for *Lemma B.2 (Box Introduction)*, an arbitrary scenario where $\vec{\mathcal{V}}(\vec{V}_2)$, i.e., $\vec{V}_2$ is an interpreted initial segment. We need to prove $t_2([(\forall z \in x)(z \in y) \wedge (\forall z \in y)(z \in x)] \to x = y)$. The formula inside t_2 is bounded, so by the *Lemma L.6 (Bounded Quantifiers Lemma)* this holds iff

$$[(\forall z \mid z \underset{2}{\in} \rho_2(\ulcorner x \urcorner)(z \underset{2}{\in} \rho_2(\ulcorner y \urcorner)) \wedge (\forall z \mid z \underset{2}{\in} \rho_2(\ulcorner y \urcorner))(z \underset{2}{\in} \rho_2(\ulcorner x \urcorner))] \to \rho_2(\ulcorner x \urcorner)) = \rho_2(\ulcorner y \urcorner)) \qquad (M1)$$

By part 2 of the definition of initial segment (*Definition A.2*), any z that is $\in_2$ a set in V_2 is a set in V_2. So the truth of *(M1)* follows by the fact that holds in initial segments. The conclusion follows by *Lemma B.2 (Box Introduction)*. ∎

M.4 Union

Proposition M.3 (Potentialist Union). $t(\forall z \, \exists a \, (\forall y \in z)(\forall x \in y)(x \in a))$

To prove this we first show that Union holds *within* any initial segment.

Lemma M.3. *If $\vec{V}$ is an interpreted initial segment and z is a variable in the language of set theory, then there is a set a in V such that* $(\forall y \in \rho(\ulcorner z \urcorner))(\forall x \in y)(x \in a)$

Proof. Suppose $\vec{V}$ is as in the statement of the lemma. Applying *Axiom 8.4 (Simple Comprehension)* we can derive

$$\Diamond_{\vec{V}}\,(\forall x)[H(x) \leftrightarrow (\exists y)(x \in y \wedge y \in \rho(\ulcorner z \urcorner))]$$

Using *Proposition B.1 (Inner Diamond)* we enter this $\Diamond_{\vec{V}}$ context . By there must be some ordinal o such that $@(\rho(\ulcorner z \urcorner), o)$. By it follows that every k satisfying $H(k)$ is available at some ordinal $o' < o$. Thus by we can infer that there is some set a whose members are just those who satisfy H. Thus, we have

$$(\exists a \mid \mathrm{set}(a))(\forall y \in \rho(\ulcorner z \urcorner)))(\forall x \in y)(x \in a)$$

This claim is implicitly content restricted to $\vec{V}$ (as by we may restrict quantifiers bounded by sets in V to sets in V. So we can leave this $\Diamond_{\vec{V}}$ context and apply *Axiom 8.2 (Diamond Elimination)* to export the conclusion. Thus completing the proof[27]. ■

Now we prove the translation of union holds

Proof. By the *Lemma M.1 (Translation Equivalence Lemma)* it suffices to prove

$$\Box(\vec{\mathcal{V}}(\overrightarrow{V_1}) \rightarrow \Diamond_{\overrightarrow{V_1}}[\overrightarrow{V_2} \underset{\mathrm{a}}{\geq} \overrightarrow{V_1} \wedge t_2((\forall y \in z)(\forall x \in y)(x \in a))])$$

I will argue by *Lemma B.2 (Box Introduction)*. So consider an arbitrary interpreted initial segment $\vec{V}_1$. By *Lemma M.3* (the above lemma) we have a 'union set' for $\rho_1(\ulcorner z \urcorner)$ in the sense of $\vec{V}_1$, i.e.,

$$(\exists a \mid \mathrm{set}_1(a))\left(\forall y \underset{1}{\in} \rho_1(\ulcorner z \urcorner))\right)\left(\forall x \underset{1}{\in} y\right)\left(x \underset{1}{\in} a\right)$$

By applying *Lemma L.1 (Pointwise Tweaking)* to the above formula we can infer that possibly ($\Diamond_{\vec{V}_1}$) we have $\vec{V}_2$ which assigns $\rho_2(\ulcorner a \urcorner)$ to this set, i.e.

$$\Diamond_{\vec{V}_1}\left[\vec{V}_2 \underset{a}{\geq} \vec{V}_1 \wedge \left(\forall y \underset{1}{\in} \rho_1(\ulcorner z \urcorner))\right)\left(\forall x \underset{1}{\in} y\right)\left(x \underset{1}{\in} \rho_2(\ulcorner a \urcorner)\right)\right]$$

Enter this $\Diamond_{\vec{V}_1}$ context. As $\vec{V}_2 \geq_a \vec{V}_1$ we may replace every $\in_1$ with $\in_2$. Since $\rho_1(\ulcorner z \urcorner) = \rho_2(\ulcorner z \urcorner)$ we may also replace $\rho_1(\ulcorner z \urcorner)$ with $\rho_2(\ulcorner z \urcorner)$ giving us

$$\left(\forall y \underset{2}{\in} \rho_2(\ulcorner z \urcorner))\right)\left(\forall x \underset{2}{\in} y\right)\left(x \underset{2}{\in} \rho_2(\ulcorner a \urcorner)\right)$$

By the *Lemma L.6 (Bounded Quantifiers)* we have

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[27] Note that we can now remove any restrictions we placed on bounded quantifiers to export the conclusion.

$$t_2((\forall y \in z)(\forall x \in y)(x \in a))])$$

Putting this together with the fact that $\overrightarrow{V_2} \geq_a \overrightarrow{V_1}$ and leaving the $\Diamond_{\overrightarrow{V_1}}$ context we have

$$\Diamond_{\overrightarrow{V_1}} [\overrightarrow{V_2} \underset{a}{\geq} \overrightarrow{V_1} \wedge t_2((\forall y \in z)(\forall x \in y)(x \in a))])$$

Discharging the assumption that $\vec{\mathcal{V}}(\vec{V}_1)$ and applying *Lemma 4.3* yields the desired result. ■

M.5 Comprehension

Proposition M.4 (Comprehension). *If θ is a formula in the language of set theory with free variables $x, w_1, \ldots, w_n$. Then:*

$$t(\forall z \forall w_1 \forall w_2 \ldots \forall w_n \exists y \forall x[x \in y \Leftrightarrow (x \in z \wedge \theta))$$

Proof. By *Lemma M.1 (Translation Equivalence Lemma)* it suffices to prove the following:

$$\begin{aligned}
&\Box[\mathcal{V}(\vec{V}_{n+1}) \rightarrow \\
&\quad \Diamond_{\vec{V}_{n+1}} (\vec{V}_{n+2} \underset{y}{\geq} \vec{V}_{n+1} \wedge \\
&\qquad \Box_{\vec{V}_{n+2}} [\quad \vec{V}_{n+3} \underset{x}{\geq} \vec{V}_{n+2} \rightarrow \\
&\qquad\qquad \rho_{n+3}(x) \underset{n+3}{\in} \rho_{n+3}(y) \leftrightarrow (\rho_{n+3}(x) \underset{n+3}{\in} \rho_{n+3}(z) \wedge t_{n+3}(\theta))])]
\end{aligned}$$

We will argue by *Lemma B2 (Box Introduction)*. So consider an arbitrary scenario where $\vec{V}_{n+1}$, is an interpreted initial segment.

We must now establish the possibility of a $\vec{V}_{n+2}$ satisfying the above. Note that $\rho_{n+2}(\ulcorner y \urcorner)$ plays the role of the set whose existance we are trying to establish.

M.5.1 Construction

To this end we will define a predicate P which applies to just those sets that belong in $\rho_{n+2}(\ulcorner y \urcorner)$.

First, we will use *Axiom 8.9 (Modal Comprehension)* to show that a predicate P could apply to exactly those objects which we want to be elements of comprehension set $\rho_{n+2}(\ulcorner y \urcorner)$. If we could quantify-in we would define $P(x)$ to hold just if (where $t^*_{n+3}(\psi)$ is $t_{n+3}(\psi)[\vec{V}_{n+3}/\vec{V}*]$)

$$x \underset{n+1}{\in} \rho_{n+1}(\ulcorner z \urcorner) \wedge \Diamond_{V_{n+1}} (\vec{V}^* \underset{x}{\geq} \vec{V}_{n+1} \wedge \rho^*(\ulcorner x \urcorner) = x \wedge t^*_{n+3}(\theta))$$

Instead we apply the *Axiom 8.9 (Modal Comprehension)* with Ψ as $\vec{\mathcal{V}}(\vec{V}_{n+1})$, R as P, $\mathcal{L}$ as $\vec{V}_{n+1}$, and ϕ as below, giving us the following.

$$\begin{aligned}
&\Diamond_{\vec{V}_{n+1}} \vec{\mathcal{V}}(\vec{V}_{n+1}) \wedge \Box_{\vec{V}_{n+1},P} [\quad (\exists! x \mid Q(x)) \rightarrow (\exists x \mid Q(x))[P(x) \leftrightarrow \vec{V}_{n+1}(x) \wedge \Phi])] \\
&\qquad\qquad \text{where} \quad \Phi = \Diamond_{\vec{V}_{n+1},Q} [(\exists! q \mid Q(q))(q \underset{n+1}{\in} \rho_{n+1}(\ulcorner z \urcorner) \wedge \\
&\qquad\qquad\qquad \left(\vec{V}^* \underset{x}{\geq} \vec{V}_{n+1} \wedge \rho^*(\ulcorner x \urcorner) = q \wedge t^*_{n+3}(\theta)\right] \qquad (M2)
\end{aligned}$$

Enter this $\Diamond_{\vec{V}_{n+1}}$ context. To show there's a set y in V_{n+1} of all the objects satisfying P, we must show that all these objects are available at layers below some layer o in V_{n+1}.

To this end we argue that

$$(\forall k)\left(P(k) \to k \underset{n+1}{\in} \rho_{n+1}(\ulcorner z \urcorner)\right) \qquad (M3)$$

Suppose the claim failed. Then by *Lemma H.7 (Multiple Definitions)*, *Axiom 8.4 (Simple Comprehension)* and *Proposition 8.1(Simplified Choice)* we can ($\Diamond_{\vec{V}_{n+1},P}$) have Q apply to a unique counterexample, i.e.,

$$(\exists!\, x \mid Q(x))(P(x) \land \neg x \underset{n+1}{\in} \rho_{n+1}(\ulcorner z \urcorner))$$

Enter this $\Diamond_{\vec{V}_{n+1},P}$ context and import the $\Box_{\vec{V}_{n+1},P}$ claim inside equation *(M2)*. Apply *Lemma B.3 (Box Elimination)* and modus ponens to infer that Φ holds. Applying *Axiom 8.2 (Diamond Elimination)* to Φ lets us infer

$$(\exists!\, q \mid Q(q))(q \underset{n+1}{\in} \rho_{n+1}(\ulcorner z \urcorner)$$

as this sentence is content restricted to $Q, \vec{V}_{n+1}$. This contradicts the assumption above. Exporting this contradiction gives the desired result.

Thus, equation *(M3)* holds. Since $\rho_{n+1}(\ulcorner z \urcorner)$ is a set in V_{n+1} by there is some ordinal level o in V_{n+1} at which $\rho_{n+1}(\ulcorner z \urcorner)$ is available and every x such that $P(x)$ occurs at some ordinal level o' below o. So, by , it follows that there's a set y in V_{n+1} available at level o, whose elements are exactly the x such that $P(x)$. So by *Lemma L.1 (Pointwise Tweaking)* we may set $\rho_{n+2}(\ulcorner y \urcorner)$ to be the set specified by P. That is

$$\Diamond_{\vec{V}_{n+1},P}\left[\vec{V}_{n+2} \underset{y}{\geq} \vec{V}_{n+1} \land (\forall x)(x \underset{n+1}{\in} \rho_{n+2}(\ulcorner y \urcorner) \leftrightarrow P(x)) \land V_{n+2} = V_{n+1}\right] \quad (M4)$$

Using *Axiom 8.6 (Importing)* we may import both equation *(M2)* and *(M3)* then apply Diamond Collapsing (*Lemma B.8*) leave us in only a single $\Diamond_{\vec{V}_{n+1}}$ context.

M.5.2 Verification

Enter this $\Diamond_{\vec{V}_{n+1}}$ context.

We need to check $\vec{V}_{n+2}$ behaves as desired. That is, we need to show that

$$\vec{V}_{n+2} \underset{y}{\geq} \vec{V}_{n+1} \land \Box_{\vec{V}_{n+2}}[\vec{V}_{n+3} \underset{x}{\geq} \vec{V}_{n+2} \to$$
$$\rho_{n+3}(x) \underset{n+3}{\in} \rho_{n+3}(y) \leftrightarrow (\rho_{n+3}(x) \underset{n+3}{\in} \rho_{n+3}(z) \land t_{n+3}(\theta))]$$

Since we already have $\vec{V}_{n+2} \geq_y \vec{V}_{n+1}$ it is enough to show that (by the definition of t_n)

$$\Box_{\vec{V}_{n+2}}\left[\vec{V}_{n+3} \underset{x}{\geq} \overrightarrow{V_{n+2}} \to t_{n+3}(x \in y \leftrightarrow x \in z \land \theta))\right]$$

By *Lemma H.1 (Box Ignoring)* (since the sentence inside the $\Box_{\vec{V}_{n+2}}$ above is content restricted to $\vec{V}_{n+2}, \vec{V}_{n+3}$) it suffices to prove the $\Box_{\vec{V}_{n+2},\vec{V}_{n+1},P}$ verson of this claim. We will argue by *Lemma B.2 (Box Introduction)*. Consider an arbitrary scenario in which $\vec{V}_{n+3} \geq_{\mathrm{x}} \vec{V}_{n+2}$ we will prove the following using only assumptions content restricted to $\vec{V}_{n+1}, P, \vec{V}_{n+2}$ (which, as the interior of equation *(M4)* is so restricted happens automatically).

$$t_{n+3}(x \in y) \leftrightarrow t_{n+3}(x \in z \wedge \theta) \qquad (M5)$$

By *Axiom 8.4 (Simple Comprehension)* it's possible ($\Diamond_{\vec{V}_{n+1},\vec{V}_{n+2},\vec{V}_{n+3},P}$) that Q selects exactly $\rho_{n+3}(\ulcorner x \urcorner)$). That is

$$(\forall k)(Q(k) \leftrightarrow k = \rho_{n+3}(\ulcorner x \urcorner))$$

Note that we may pull in all facts in the current context by operation of *Axiom 8.4 (Simple Comprehension)*. Enter this $\Diamond_{\vec{V}_{n+1},\vec{V}_{n+2},\vec{V}_{n+3},P}$ context. We now argue that equation *(M5)* holds in this context.

Note that

$$t_{n+3}(x \in y) \overset{\text{def}}{\leftrightarrow} \rho_{n+3}(\ulcorner x \urcorner) \underset{n+3}{\in} \rho_{n+3}(\ulcorner y \urcorner)$$

By the fact that $V_{n+3} \geq_x V_{n+2} \geq_y V_{n+1}$ and our specification of $\rho_{n+2}(\ulcorner y \urcorner)$, c.f. *(M3)*, we have

$$\rho_{n+3}(\ulcorner x \urcorner) \underset{n+3}{\in} \rho_{n+3}(\ulcorner y \urcorner) \leftrightarrow P(\rho_{n+3}(\ulcorner x \urcorner))$$

Since Q applies uniquely to $\rho_{n+3}(\ulcorner x \urcorner)$) applying *Lemma B.3 (Box Elimination)* to equation *(M2)* and invoking modus ponens we may infer

$$P\big(\rho_{n+3}(\ulcorner x \urcorner)\big) \leftrightarrow \vec{V}_{n+1}\big(\rho_{n+3}(\ulcorner x \urcorner)\big) \wedge \Phi. \qquad (M6)$$

where

$$\Phi = \Diamond_{\vec{V}_{n+1},Q} \,(\exists!\, q \mid Q(q))(\psi(q))$$

And

$$\psi(q) \overset{\text{def}}{\leftrightarrow} \mathrm{q} \underset{n+1}{\in} \rho_{n+1}(\ulcorner z \urcorner) \wedge \vec{V}^{*} \geq_{\mathrm{x}} \vec{V}_{n+1} \wedge \rho^{*}(\ulcorner x \urcorner) = q \wedge t^{*}_{n+3}(\theta)$$

As the sentence inside $\Diamond_{\vec{V}_{n+1},Q}$ in Φ is content restricted to $\vec{V}_{n+1}, \vec{V}^{*}, Q$ by $\Diamond$ Ignoring (*Axiom 8.3*)

$$\Phi \leftrightarrow \Diamond_{\mathcal{L}} \,(\exists!\, q \mid Q(q))(\psi(q))$$

where $\mathcal{L} = \{\vec{V}_{n+1}, \vec{V}_{n+2}, \vec{V}_{n+3}, P, Q\}$. Hence the right side of equation *(M6)* holds iff

$$\vec{V}_{n+1}(\rho_{n+3}(\ulcorner x \urcorner)) \wedge \Diamond_{\mathcal{L}} \,(\exists!\, q \mid Q(q))(\psi(q))$$

Using *Axiom 8.6 (Importing)* to import the fact that Q applies uniquely to $\rho_{n+3}(\ulcorner x \urcorner)$ into the $\Diamond_{\mathcal{L}}$ context above (and applying *Axiom 8.7 Logical Closure*) we may infer that the above formula is equivalent to (the reverse direction also follows by importing the same fact into the $\Diamond_{\mathcal{L}}$ context below)

$$\vec{V}_{n+1}(\rho_{n+3}(\ulcorner x \urcorner)) \wedge \Diamond_{\mathcal{L}}(\psi(\rho_{n+3}(\ulcorner x \urcorner))) \qquad (M7)$$

We further observe that $\psi(\rho_{n+3}(\ulcorner x \urcorner))$ implies that $\rho_{n+3}(\ulcorner x \urcorner) \in_{n+1} \rho_{n+1}(\ulcorner z \urcorner)$ and hence $\vec{V}_{n+1}(\rho_{n+3}(\ulcorner x \urcorner))$. By *Axiom 8.2 (Diamond Elimination)* it follows that we can derive $\vec{V}_{n+1}(\rho_{n+3}(\ulcorner x \urcorner))$ from $\Diamond_{\mathcal{L}}(\psi(\rho_{n+3}(\ulcorner x \urcorner)))$. Hence, the above equation *(M7)* is equivalent to

$$\Diamond_{\mathcal{L}}(\psi(\rho_{n+3}(\ulcorner x \urcorner)))$$

Thus $t_{n+3}(x \in y)$ is true iff

$$\begin{aligned} \Diamond_{\mathcal{L}}[\quad & \rho_{n+3}(\ulcorner x \urcorner) \underset{n+1}{\in} \rho_{n+1}(\ulcorner z \urcorner) \wedge \\ & \vec{V}^{*} \underset{x}{\geq} \vec{V}_{n+1} \wedge \rho^{*}(\ulcorner x \urcorner) = \rho_{n+3}(\ulcorner x \urcorner) \wedge t_{n+3}^{*}(\theta)] \end{aligned} \qquad (M8)$$

It only remains to show that *(M8)* is true iff $t_{n+3}(x \in z \wedge \theta)$ where

$$t_{n+3}(x \in z \wedge \theta) \leftrightarrow t_{n+3}(x \in z) \wedge t_{n+3}(\theta)$$

.

Suppose that equation *(M8)* is true. Enter the $(\Diamond_{\mathcal{L}})$ context for this claim. We can import the fact that $\vec{V}_{n+3} \geq_x \vec{V}_{n+2} \geq_y \vec{V}_{n+1}$ and thus derive that $t_{n+3}(x \in z)$ from the first conjunct of the interior of equation *(M8)*. We now work to prove $t_{n+3}(\theta)$ by way of the *Corollary L.1 (Generalized Translation Lemma)* applied to $t_{n+3}^{*}(\theta)$. We already know that V^{*}, V_{n+3} both extend V_{n+1}. We now argue that ρ^{*} and ρ_{n+3} assign each free variable in θ to the same object in V_{n+1} as follows:

- Since $\vec{V}^{*} \geq_x \vec{V}_{n+1}$ and $\vec{V}_{n+3} \geq_{x,y} \vec{V}_{n+1}$ we have $\rho^{*} = \rho_{n+1} = \rho_{n+3}$ for all variables names other than 'x' and 'y.'
- As $\rho^{*}(\ulcorner x \urcorner) = \rho_{n+3}(\ulcorner x \urcorner)$, ρ^{*} and ρ_{n+3} both assign 'x' to the same object in V_{n+1}.
- So $\vec{V}^{*}$ can only disagree with V_{n+3} on y, and by the assumptions of the theorem to be proved y isn't free in θ.

So by the *Corollary L.1 (Generalized Translation Lemma)* we have $t_{n+3}^{*}(\theta) \leftrightarrow t_{n+3}(\theta)$ and hence $t_{n+3}(\theta)$. As $t_{n+3}(\theta)$ is content restricted to $\vec{V}_{n+3}$ we can export this conclusion from the $\Diamond_{\mathcal{L}}$ context above.

Conversely, suppose $t_{n+3}(x \in z) \wedge t_{n+3}(\theta)$. By *Lemma L.1 (Pointwise Tweaking)* applied to (V_{n+3}, ρ_{n+1}) and $(\exists y)(y = \rho_{n+3}(\ulcorner x \urcorner))$ it is possible $(\Diamond_{\mathcal{L}})$ that $\vec{V}^{*}$ agrees with (V_{n+3}, ρ_{n+1}) except in that it assigns 'x' to $\rho_{n+3}(\ulcorner x \urcorner)$ i.e,

$$\Diamond_{\mathcal{L}} \left[\rho^*(\ulcorner x \urcorner) = \rho_{n+3}(\ulcorner x \urcorner) \wedge \vec{V}^* \underset{x}{\geq} \vec{V}_{n+1} \wedge V^* = V_{n+3}\right]$$

We now seek to derive equation *(M8)*. Enter this $\Diamond_{\mathcal{L}}$ context. The first conjunct inside the $\Diamond_L$ in equation *(M8)* follows from the (importable) fact that $t_{n+3}(x \in z)$ and the first conjunct inside the $\Diamond_{\mathcal{L}}$ above. The second conjunct inside equation *(M8)* is identical to the second conjunct inside the above equation and the third conjunct inside equation *(M8)* is equal to the first conjunct inside the above equation.

Finally, we use the *Corollary L.1 (General Translation Lemma)* to infer $t^*_{n+3}(\theta)$. From the above equation we have $V^* \geq V_{n+3}$ and $\vec{V}^* \geq_x \vec{V}_1$. We can import the fact that $\vec{V}_{n+3} \geq_{x,y} \vec{V}_{n+1}$. Thus, (as above) $\rho^*$and ρ_{n+3} both agree with ρ_{n+1} on all other variables except for y (and y isn't free in $x \in z \wedge \theta$). So, just as above, we can apply *Corollary L.1* to get $t^*_{n+3}(\theta) \leftrightarrow t_{n+3}(\theta)$ and hence derive the third conjunct needed $t^*_{n+3}(\theta)$. Thus we can leave this $\Diamond_L$ context and conclude that *(M8)* holds.

Thus, equation *(M8)* is true iff $t_{n+3}(x \in z \wedge \theta)$

This completes our verification of equation *(M5)* on the assumption that $\vec{V}_{n+3} \geq_x \vec{V}_{n+2}$. Hence,

$$\vec{V}_{n+3} \underset{x}{\geq} \overrightarrow{V_{n+2}} \rightarrow t_{n+3}(x \in y \leftrightarrow x \in z \wedge \theta))$$

As we proved this from only the assumption that the interior of equation *(M4)* was true, we may use *Lemma B.2 (Box Introduction)* to infer

$$\Box_{\vec{V}_{n+2},\vec{V}_{n+1},P} \left[\vec{V}_{n+3} \underset{x}{\geq} \overrightarrow{V_{n+2}} \rightarrow t_{n+3}(x \in y \leftrightarrow x \in z \wedge \theta))\right]$$

As remarked above we may use *Lemma H.1 (Box Ignoring)* to conclude

$$\Box_{\vec{V}_{n+2}} \left[\vec{V}_{n+3} \underset{x}{\geq} \overrightarrow{V_{n+2}} \rightarrow t_{n+3}(x \in y \leftrightarrow x \in z \wedge \theta))\right]$$

Dropping out of the $\Diamond_{\vec{V}_{n+1}}$ context and applying *Lemma B.2 (Box Introduction)* again we reach the desired conclusion.

$$\begin{aligned}
&\Box[\mathcal{V}(\vec{V}_{n+1}) \rightarrow \\
&\quad \Diamond_{\vec{V}_{n+1}} (\vec{V}_{n+2} \underset{y}{\geq} \vec{V}_{n+1} \wedge \\
&\qquad \Box_{\vec{V}_{n+2}} [\; \vec{V}_{n+3} \underset{x}{\geq} \vec{V}_{n+2} \rightarrow \\
&\qquad\quad \rho_{n+3}(x) \underset{n+3}{\in} \rho_{n+3}(y) \leftrightarrow (\rho_{n+3}(x) \underset{n+3}{\in} \rho_{n+3}(z) \wedge t_{n+3}(\theta))])]
\end{aligned}$$

■

M.6 Pairing

Proposition M.1 (Potentialist Pairing). $t(\forall x \forall y \exists z(x \in z \wedge y \in z))$

Proof. By the *Lemma M.1 (Translation Equivalence Lemma)* it suffices to prove the following:

$$\Box(\vec{\mathcal{V}}(\vec{V}_2) \rightarrow \Diamond_{\vec{V}_2}\left[\vec{V}_3 \underset{z}{\geq} \vec{V}_2 \wedge t_3(x \in z \wedge y \in z)\right])$$

Consider an arbitrary interpreted initial segment $\vec{V}_2$. We will show that it's logically possible to extend this segment with another initial segment containing a set z which has both $\rho_2(\ulcorner x \urcorner)$ and $\rho_2(\ulcorner y \urcorner)$ as members and then invoke *Lemma B.2 (Box Introduction)* to derive the desired conclusion.

By the Proper Extension Lemma (*Lemma K.4*), it's possible ($\Diamond_{\vec{V}_2}$) to have $\vec{V}_3$ properly extend $\vec{V}_2$.

$$\Diamond_{\vec{V}_2}[\vec{V}_3 \geq \vec{V}_2 \wedge (\exists o)(\mathrm{ord}_3(o) \wedge (\forall u)(\mathrm{ord}_2(u) \rightarrow u \underset{3}{<} o)]$$

Enter this $\Diamond_{\vec{V}_2}$ context. I claim $\vec{V}_3$ contains a set z which has exactly $\rho_2(\ulcorner x \urcorner)$ and $\rho_2(\ulcorner y \urcorner)$ as members, i.e., $(\forall k)(k \in_3 z \leftrightarrow k = \rho_2(\ulcorner x \urcorner) \vee k = \rho_2(\ulcorner y \urcorner))$. By Simple Comprehension (*Axiom 8.4*) it's possible ($\Diamond_{\vec{V_2},\vec{V}_3}$) to maintain all the facts we have so far and for H to apply to exactly this pair of objects, i.e.,

$$(\forall k)[H(k) \leftrightarrow k = \rho_2(\ulcorner x \urcorner) \vee k = \rho_2(\ulcorner y \urcorner)]$$

Enter this $\Diamond_{\vec{V_2},V_3}$ context. Note that all x such that $H(x)$ are in $Ext(V_2)$. Thus for each such u we have $@(x,u)$ for some $\mathrm{ord}_2\ u$. But by the fact that V_3 properly extends V_2 and by part there is a $set_3\ k$ whose elements are exactly $\rho_2(x)$ and $\rho_2(y)$. So we have

$$(\exists z)[set_3(z) \wedge (\forall k)(k \underset{3}{\in} z \leftrightarrow k = \rho_2(\ulcorner x \urcorner) \vee \rho_2(\ulcorner y \urcorner))]$$

We now apply *Lemma M.1* to (V_3, ρ_2), to get the possibility ($\Diamond_{\vec{V_2},V_3,H}$) of a ρ_3 such that $\vec{V}_3 \geq_x \vec{V}_2$ which chooses $\rho_3(\ulcorner z \urcorner)$ to witness the truth of the above existential claim. This yields:

$$\Diamond_{\vec{V_2},V_3,H}\left[\vec{V_3} \underset{z}{\geq} \vec{V_2}(\forall k)(k \underset{3}{\in} \rho_3(\ulcorner z \urcorner) \leftrightarrow k = \rho_2(\ulcorner x \urcorner) \vee \rho_2(\ulcorner y \urcorner)\right]$$

Enter this $\Diamond_{\vec{V_2},V_3,H}$ context. Unpacking definitions (note that $\vec{V_3} \geq_z \vec{V_2}$ implies thatρ_3 agrees with ρ_2 on $\ulcorner x \urcorner$ and $\ulcorner y \urcorner$) we can infer:

$$\rho_3(\ulcorner x \urcorner) \underset{3}{\in} \rho_3(\ulcorner z \urcorner) \wedge \rho_3(\ulcorner y \urcorner) \underset{3}{\in} \rho_3(\ulcorner z \urcorner)]$$

Hence

$$\vec{V_3} \underset{z}{\geq} \vec{V_2} \wedge t_3(x \underset{3}{\in} z \wedge y \underset{3}{\in} z)$$

So successively exiting all the $\Diamond$ contexts we entered yields

$$\Diamond_{\vec{V_2}}\ \Diamond_{\vec{V_2},V_3}\ \Diamond_{\vec{V_2},V_3,H}\ \Diamond_{\vec{V_2},V_3,H,\rho_3}\left[\vec{V}_3 \underset{z}{\geq} \vec{V}_2 \wedge t_3(x \in z \wedge y \in z)\right])$$

Finally, applying *Lemma H.6 (Diamond Simplification)* yields the desired $\Diamond_{\vec{V}_2}$ claim. This completes our *Lemma B.2 (Box Introduction)* argument giving us our desired result.

$$\Box(\vec{\mathcal{V}}(\overrightarrow{V_2}) \to \Diamond_{\overrightarrow{V_2}} \left[\vec{V}_3 \underset{z}{\geq} \vec{V}_2 \wedge t_3(x \in z \wedge y \in z)\right])$$

■

M.7 Powerset

Proposition M.5 (Potentialist Powerset). $t(\forall x \exists y \forall z[(\forall w)(w \in z \to w \in x) \to z \in y])$

We first prove the trivial fact that the elements of the set $\vec{V}_{n+1}$ assigns to the variable x are all sets in V_{n+1}.

Lemma M.4. *If* $V_{n+1} \geq V_n$ *and* $\vec{\mathcal{V}}(\vec{V}_n)$ *then* $(\forall b)[\mathrm{set}_{n+1}(b) \wedge (\forall c \underset{n+1}{\in} b)(c \underset{n+1}{\in} \rho_n(\ulcorner x \urcorner)) \to \mathrm{set}_n(b)]$.

Proof. Suppose, for contradiction, that the assumptions of the lemma hold but the lemma fails. Then there is a counterexample.

$$(\exists b)\left[\mathrm{set}_{n+1}(b) \wedge \left(\exists c \mid c \underset{n+1}{\in} b\right)\left[c \underset{n+1}{\in} \rho_n(\ulcorner x \urcorner)\right] \wedge \neg \mathrm{set}_n(c)\right]. \qquad (M9)$$

By *Lemma H.7 (Multiple Definitions)* , using *Axiom 8.4 (Simple Comprehension)* and *Proposition 8.1 (Simple Choice)* it is possible ($\Diamond_{\vec{V}_{n+1},\vec{V}_n}$) for all our assumptions to remain true while B, K apply to a unique objects b, c witnessing equation *(M9)*.

Enter this $\Diamond_{\vec{V}_{n+1},\vec{V}_n}$ context. Putting all these conditions above together we can derive contradiction by straightforwardly applying the definitions of interpreted initial segment (*Definition A.4*) and extension (*Definition A.3*) using the fact that $\in_{n+1}$ agrees with $\in_n$ on sets in V_n. ■

We now prove the desired result.

Proof. By the *Lemma M.1 (Translation Equivalence Lemma)* it suffices to prove the following:

$$\Box[\quad \vec{\mathcal{V}}(\vec{V}_1) \to \Diamond_{\vec{V}_1} (\vec{V}_2 \underset{y}{\geq} \vec{V}_1 \wedge \Box_{\vec{V}_2} [\vec{V}_3 \underset{z}{\geq} \vec{V}_2 \to t_3(\phi(z,x,y))])]$$

where
$\phi(z, x)$ abbreviates $(\forall w \in z)(w \in x) \to z \in y$

We will argue by *Lemma B.2 (Box Introduction)*. So consider an arbitrary situation in which $\vec{\mathcal{V}}(\overrightarrow{V_1})$.

We seek to show the possibility of an initial segment extending V_1 containing the powerset of $\rho_1(\ulcorner x \urcorner)$. The first three steps of our proof are exactly miror what we said about pairing in the previous proof.

Just as before, the *Lemma K.4 (the Proper Extension Lemma)* lets us conclude that possibly ($\Diamond_{\vec{V}_1}$) we can have V_2 properly extend V_1, i.e.,

$$\lozenge_{\vec{V}_1}\left[V_2 \geq V_1 \wedge (\exists o)\left(\mathrm{ord}_2(o) \wedge (\forall u)\left(\mathrm{ord}_1(u) \rightarrow u \underset{2}{<} o\right)\right)\right] \qquad (M10)$$

Enter this $\lozenge_{\vec{V}_1}$ context and import the fact that $\vec{\mathcal{V}}(\overrightarrow{V_1})$. We will show that this V_2 contains a set a (our intended powerset) whose elements are exactly those sets b in V_2 such that $b \subset_2 \rho_1(\ulcorner x \urcorner)$. Applying Simple Comprehension (*Axiom 8.4*) shows that H could apply (while maintaining truth of equation *(M10)* inside the $\lozenge_{\vec{V}_1}$) to exactly the intended elements of our set a. That is :

$$\lozenge_{\vec{V}_1,V_2}\,(\forall b)\left(H(b) \leftrightarrow \mathrm{set}_2(b) \wedge (\forall c \mid c \underset{2}{\in} b)(c \underset{2}{\in} \rho_1(\ulcorner x \urcorner))\right)$$

Enter this $\lozenge_{\vec{V}_1,V_2}$ context. By the fact that $\vec{\mathcal{V}}(\vec{V}_1)$, we know that $\rho_1(\ulcorner x \urcorner)$ occurs at some level u in V_1. Now consider an arbitrary b in the extension of H. All elements of b are $\in_2 \rho_1(\ulcorner x \urcorner)$. So by *Lemma M.4 (Powerset Helper)*, b is available at some level u in V_1.

By the fact that V_2 properly extends V_1, it contains a level o such that $o >_2 u$ for all u such that $\mathrm{ord}_1(u)$. So all objects in the extension of H occur at levels below o. So by the fact that $\mathcal{V}(V_2)$ (specifically) there's a $\mathrm{set}_2(k)$ (occurring at o) coextensive with H. Hence, there's a set_2 whose members are exactly those objects satisfying the right-hand side of the biconditional characterizing H above.

Applying *Lemma L.1 (Pointwise Tweaking)* to V_2, ρ_1 and importing all relevant facts establishes the possibility ($\lozenge_{\overrightarrow{V_1},V_2}$) of a ρ_2 such that $\vec{V}_2 \geq_y (V_2, \rho_1)$ which assigns $\rho_2(\ulcorner y \urcorner)$ to witness the truth of the above existential claim.

$$\lozenge_{\vec{V}_1,V_2,H}\,[\vec{V}_2 \underset{y}{\geq} \vec{V}_1 \wedge (\forall a)(\quad b \underset{2}{\in} \rho_2(\ulcorner y \urcorner) \leftrightarrow$$
$$\mathrm{set}_2(b) \wedge (\forall c \mid c \underset{2}{\in} b)(c \underset{2}{\in} \rho_1(\ulcorner x \urcorner))]$$

Enter this $\lozenge_{\vec{V}_1,\vec{V}_2,H}$ context. We have $\vec{V}_2 \geq_y \vec{V}_1$ immediately. So it just remains to show that $\square_{V_2}$ $[\vec{V}_3 \geq_z \vec{V}_2 \rightarrow t_3(\phi(z,x,y))]$, i.e., to prove the *potentialistic translation* of the claim that $\rho_2(\ulcorner y \urcorner)$ contains all subsets Z of $\rho_1(\ulcorner x \urcorner)$.

By the fact that $\vec{V}_2 \geq_y \vec{V}_1$ and thus $\rho_2(\ulcorner x \urcorner) = \rho_1(\ulcorner x \urcorner)$, we can infer the following (which is content restricted to $\vec{V}_2$)

$$(\forall b)\left(b \underset{2}{\in} \rho_2(\ulcorner y \urcorner) \leftrightarrow \mathrm{set}_2(b) \wedge \left(\forall c \,\middle|\, c \underset{2}{\in} b\right)\left(c \underset{2}{\in} \rho_2(\ulcorner x \urcorner)\right)\right) \qquad (M11)$$

Now, for application of *Lemma B.2 (Box Introduction)*, assume that $\vec{V}_3 \geq_z \vec{V}_2$ while holding fixed the facts about $\vec{V}_2$. As we wish to infer a $\square_{\vec{V}_2}$ claim we may use equation *(M11)* in our derivation. By the definition of $\geq_z$ we can infer

$$(\forall b)(b \underset{3}{\in} \rho_3(\ulcorner y \urcorner) \leftrightarrow \mathrm{set}_3(b) \wedge (\forall c \mid c \underset{3}{\in} b)(c \underset{3}{\in} \rho_3(\ulcorner x \urcorner)))$$

Instantiating b with $\rho_3(\ulcorner z \urcorner)$ (and noting that clearly $\text{set}_3(\rho_3(\ulcorner z \urcorner))$ lets us deduce $\phi^{\vec{V}_3}$, i.e.,

$$(\forall w \underset{3}{\in} \rho_3(\ulcorner z \urcorner)(w \underset{3}{\in} \rho_3(\ulcorner x \urcorner)) \rightarrow \rho_3(\ulcorner z \urcorner) \underset{3}{\in} \rho_3(\ulcorner y \urcorner)$$

Applying the *Lemma L.1* lets us infer $t_3(\phi(z,x,y))$. This completes our *Lemma B.2 (Box Introduction)* argument letting us deduce

$$\Box_{\vec{V}_2}[\vec{V}_3 \underset{z}{\geq} \vec{V}_2 \rightarrow t_3(\phi(z,x,y))]$$

.

Now exiting all our $\Diamond$ contexts yields

$$\Diamond_{\vec{V}_1} \Diamond_{\vec{V}_1,V_2} \Diamond_{\vec{V}_1,V_2,H}(\vec{V}_2 \underset{y}{\geq} \vec{V}_1 \wedge \Box_{\vec{V}_2}[\vec{V}_3 \underset{z}{\geq} \vec{V}_2 \rightarrow t_3\big(\phi(z,x,y)\big)])$$

By Diamond Collapsing (*Lemma B.8*) it follows that

$$\Diamond_{\vec{V}_1}(\vec{V}_2 \underset{y}{\geq} \vec{V}_1 \wedge \Box_{\vec{V}_2}[\vec{V}_3 \underset{z}{\geq} \vec{V}_2 \rightarrow t_3\big(\phi(z,x,y)\big)])$$

And, as we have proved this from only the assumption that our original context satisfies $\mathcal{V}(\vec{V}_1)$, the theorem follows by a final application of *Lemma B.2 (Box Introduction)*.

$$\Box[\quad \vec{\mathcal{V}}(V_1) \rightarrow \Diamond_{\vec{V}_1}(\vec{V}_2 \underset{y}{\geq} \vec{V}_1 \wedge \Box_{\vec{V}_2}[\vec{V}_3 \underset{z}{\geq} \vec{V}_2 \rightarrow t_3(\phi(z,x,y))])]$$

■

M.8 Choice

Definition M.2 (Choice Function). *We first adopt the following standard notation from set theory.*

Definition M.1 *(Ordered Pair).* (y,z) *denotes the unique set* $\{\{y\},\{y,z\}\}$ *and* $w=(y,z)$ *denotes the (bounded) formula*

$(\exists a \quad, b \in w)(\forall q \in w)([q = a \vee q = b] \wedge$
$\quad y \in a \wedge y \in b \wedge z \in b \wedge (\forall o \in a)(o = y) \wedge (\forall o \in b)(o = z \vee o = a)$

$\text{CH}(f,x)$ *abbreviates the set theoretic sentence which claims that set* f *is a choice function for set* x. *That is* f *associates to each element* y *of* x *a unique element of* y, *i.e.,* $\text{CH}(f,x) \overset{\text{def}}{\leftrightarrow} (\forall y \mid y \in x)(f(y) \in y)$

Note that $f(y) \in y$ iff $(\exists w \in f)(\exists z \in y)(w = (y,z))$ so $\text{CH}(f,x)$ is a bounded formula.

Proposition M.6 (Potentialist Choice). $t(\forall x[\emptyset \notin x \rightarrow (\exists f)\text{CH}(f,x)])$

Before we prove this result, we first note the following lemmas.

Lemma M.5. *If $V'' \gneq V' \gneq V$ (where $\gneq$ indicates a proper extension) and y, z are sets in V then there is some set w in V'' equal to (y, z).*

Proof. The reasoning in the proof of *Proposition M.5 (Potentialist Pairing)* implies that $\{y, z\}$ is a set in V' and a similar argument implies that $\{y\}$ is as well. Applying that reasoning again implies that $\{\{y\}, \{y, z\}\}$ is a set in V'' ■

Lemma M.6. *Suppose that $R^*(x, y)$ is a function (i.e. relation taking each x to a unique y) taking sets in V to sets in V and $V''' \gneq V'' \gneq V' \gneq V$ (where $\gneq$ indicates a proper extension) then there is a set f in V''' which represents the set theoretic function defined by $R^*(x, y)$, i.e., $(\forall a)(\forall b)[(a, b) \in '''f \leftrightarrow R^*(a, b)]$*

Proof. This follows straightforwardly from *Lemma M.5 (Ordered Pair Existence)* by using Simple Comprehension (*Axiom 8.4*), the as well as the fact that V''' properly extends V'' and then exporting the conclusion. ■

Our strategy to prove *Proposition M.6 (Potentialist Choice)* is as follows. Given some interpreted initial segment $\vec{V}_1$ our strategy will be to define I to apply to the elements of $\rho_1(\ulcorner x \urcorner)$ and define R to relate $R(y, z)$ just when $y \in \rho_1(\ulcorner x \urcorner)$ and $z \in y$. We will then use *Axiom 8.12 (Choice)* to infer the possibility of a choice relation R^*. We will then use the above lemmas to derive the possibility of V_2 containing a set f of ordered pairs coding the function given by R^*. This set f will satisfy $\mathrm{CH}(f, x)$ allowing us to invoke the *Lemma L.1 (Bounded Quantifiers)* to establish the desired translation.

Proof. By the *Lemma M.1 (Translation Equivalence Lemma)* it suffices to prove the following:

$$\Box[\mathcal{V}(\vec{V}_1) \wedge t_1(\emptyset \notin x) \rightarrow \Diamond_{\vec{V}_1} (\vec{V}_2 \underset{f}{\geq} \vec{V}_1 \wedge t_2(\mathrm{CH}(f, x))]$$

To prove this by *Lemma B.2 (Box Introduction)* suppose that $\mathcal{V}(\vec{V}_1) \wedge t_1(\emptyset \notin x)$. We now prove $\Diamond_{\vec{V}_1} (\vec{V}_2 \geq_f \vec{V}_1 \wedge t_2(\mathrm{CH}(f, \ulcorner x \urcorner))$.

Note that by the *Lemma L.6 (Bounded Quantifiers)* it suffices to show

$$\Diamond_{\vec{V}_1} (\vec{V}_2 \underset{f}{\geq} V_1 \wedge [\mathrm{CH}(\rho_2(\ulcorner f \urcorner), \rho_2(\ulcorner x \urcorner))]^{V_2})$$

By the Multiple Definitions Lemma (*Lemma H.7*) it's possible ($\Diamond_{\overrightarrow{V_1}}$) for our assumption $\mathcal{V}(\vec{V}_1) \wedge t_1(\emptyset \notin x)$ to remain true while defining I, R via Simple Comprehension (*Axiom 8.4*) so that

$$\begin{aligned} (\forall z)(I(z) &\leftrightarrow z \underset{1}{\in} \rho_1(\ulcorner x \urcorner)) \\ (\forall y)(\forall y')[R(y, y') &\leftrightarrow y \underset{1}{\in} \rho_1(\ulcorner x \urcorner) \wedge y' \in y] \end{aligned}$$

Enter this ($\Diamond_{\overrightarrow{V_1}}$) context. We now show that the antecedent of *Axiom 8.12 (Choice)* is satisfied, i.e., $(\forall y)[I(y) \rightarrow (\exists z)R(y, z)]$. To this end note that $t_1(\emptyset \notin x)$ abbreviates $t_1((\forall y)(y \in x \rightarrow$

$(\exists z)(z \in y))$ and apply the *Lemma L.6 (Bounded Quantifiers)* to infer that $[(\forall y)(y \in \rho_1(\ulcorner x \urcorner) \to (\exists z)(z \in y))]^{V_1}$, i.e.,

$$(\forall y)(y \underset{1}{\in} \rho_1(\ulcorner x \urcorner) \to (\exists z)(z \underset{1}{\in} y))$$

Combining this with the biconditionals specifying extensions for I and R above yields the desired result that $(\forall y)[I(y) \to (\exists z)R(y,z)]$

So by *Axiom 8.12 (Choice)*, it's possible ($\Diamond_{\vec{V}_1,I,R}$) (while retaining all our previous facts) that R^* codes a choice function for R, I, i.e., that R^*. is a *function* with domain I and $R^*(y,z)$ implies $R(y,z)$. Note that by applying the definitions of R and I we can easily deduce that R^* associates to each $y \in_1 x$ some $z \in_1 y$. We will use R^* to define a corresponding choice function in the sense of set theory, i.e., an f such that

$$(\forall a)(\forall b)[\langle a,b\rangle \underset{2}{\in} f \leftrightarrow R^*(a,b)]$$

Specifically, we apply the Proper Extension Lemma (*Lemma K.4*) three times followed by an application of *Lemma B.8 (Diamond Collapsing)* to establish the possibility ($\Diamond_{\vec{V}_1,R,I,R^*}$) of V_2 properly extending V'', properly extending V' in turn properly extending V_1. Now by *Lemma M.6 (Set Coding a Function)* and *Lemma L.1 (Pointwise Tweaking)* we can ($\Diamond_{\vec{V}_1,R,I,R^*,V_2}$) have $\vec{V}_2 \geq_f \vec{V}_1$ such that $\rho_2(\ulcorner f \urcorner)$ is a set k with the property that

$$(\forall a)(\forall b)[\langle a,b\rangle \underset{2}{\in} \rho_2(\ulcorner f \urcorner) \leftrightarrow R^*(a,b)]$$

Enter this $\Diamond_{\vec{V}_1,R,I,R^*,V_2}$ context and infer $[\mathrm{CH}(\rho_2(\ulcorner f \urcorner), \rho_2(\ulcorner x \urcorner))]^{V_2}$. Finally, leave the intervening contexts and apply *Lemma B.8 (Diamond Collapsing)* to yield the desired conclusion. ■

M.9 Potentialist Infinity

Proposition M.7 (Potentialist Infinity). $t((\exists x)[\emptyset \in x \wedge (\forall y \in x)(S(y) \in x)])$

where $S(y)$ is $y \cup \{y\}$[28]

Before we prove this result, we first prove the following lemma which establishes the possibility of an interpreted initial segment containing a successor closed set. Our strategy here will be to use the *Theorem K.2 (Fleshing Out)* on the well-ordering from the Infinite Well-Ordering Theorem (*Theorem J.1*) to argue for the possibility of a V_ω. We then use the Proper Extension Lemma (*Lemma K.4*) to construct the powerset of V_ω and argue that this set has the desired property.

Lemma M.7. $\Diamond\, [\mathcal{V}(V) \wedge (\exists x)(\emptyset \in x \wedge (\forall y \in x)(S(y) \in x))]$

---

[28] *That is, $S(y,z)$ abbreviates* $(\forall w \in z)[w \in y \vee w = y] \wedge y \in z \wedge (\forall w \in y)[w \in z]$

Proof. By the Infinite Well-Ordering Theorem (*Theorem J.1*) we may conclude it's logically possible (◊) that $\text{ord}_\omega, <_\omega$ form an infinite well-ordering with no maximal element as well as the other conclusions of the Infinite Well-Ordering Theorem (*Theorem J.1*). Enter this $\diamond$ context.

By the *Theorem K.2 (Fleshing Out)* it is further possible ($\lozenge_{\text{ord}_\omega, \leq_\omega}$) for V_ω to be an initial segment with ordinals $\text{ord}_\omega, \leq_\omega$. Enter this $\diamond_{\text{ord}_\omega, \leq_\omega}$ context and import all facts established so far. By the the Proper Extension Lemma (*Lemma K.4*) it is possible ($\diamond_{V_\omega}$) that V is an initial segment properly extending this V_ω. Enter this $\diamond_{V_\omega}$ context and import all facts established so far.

By the fact that V properly extends V_ω, we know that there's an $\text{ord}(u)$ such that $u > o$ for all o such that $\text{ord}_\omega(o)$. Every set x in V_ω occurs at some such level o, by the fact that $V \geq V_\omega$. So by condition (fatness) applied to the property set_ω, there's a $\text{set}(y)$ in V (available at level u) whose elements are exactly the sets in V_ω. It is now enough to show that set_ω contains $\emptyset$ and is successor closed.

$\emptyset$ clearly satisfies set_ω. To show that set_ω is successor closed suppose that it's not and, by the usual trick using Simple Comprehension (*Axiom 8.4*) and *Proposition 8.1 (Simplified Choice)* (via the the Multiple Definitions Lemma (*Lemma H.7*)), we may infer the possibility of Q applying uniquely to some x satisfying set_ω such that $S(x)$ doesn't satisfying set_ω. Via the Multiple Definitions Lemma (*Lemma H.7*) we may also suppose that P applies to those z such that $z = x$ or $z \in x$, i.e., $Q(z) \vee (\exists x)(Q(x) \wedge z \in x)$. By *Definition A.2 (Initial Segment)* it follows that x and every $z \in x$ occurs at some ordinal o satisfying $\text{ord}_\omega(o)$. As $\text{ord}_\omega, <_\omega$ has no maximal element there are ordinals o', o'' in V_ω such that $o <_\omega o' <_\omega o''$. Clearly $\{x\}$ is availible at o' and thus by the fatness condition (condition 5) in *Definition A.2 (Initial Segment)* applied to the property P it follows that $S(x)$ is available at o'' and thus satisfies set_ω giving the desired contradiction.

So we have

$$\mathcal{V}(V) \wedge (\exists x)(\emptyset \in x \wedge (\forall y \in x)(S(y) \in x))$$

The theorem follows by leaving all ◊ contexts entered and applying *Lemma B.8 (Diamond Collapsing)*. ■

With this fact in hand, we can now prove *Proposition M.7 (Potential Infinity)* as follows.

Proof. As noted above, our strategy will be to establish the possibility of a V_1 containing a set x which is successor closed and then argue that if $\rho_1(\ulcorner x \urcorner) = x$ then $t_1(\theta)$, where θ asserts that x is successor closed, is also true.

To set this up, we need to get the claim x is successor closed into a proper form for applying the bounded quantifiers lemma. So note that by *Theorem 9.1 (Logical Closure of Translation)* it suffices to prove the translation of the logically equivalent claim.

$$t((\exists x)\phi(x))$$
where
$$\phi(x) \overset{\text{def}}{\leftrightarrow} [\emptyset \in x \wedge (\forall y \in x)(\exists y' \in x)(S(y) = y')]$$
$$S(y) = y' \overset{\text{def}}{\leftrightarrow} y \in y' \wedge (\forall z \in y)(z \in y') \wedge (\forall z \in y')(z = y \vee z \in y)$$

Note that when the definition of $\emptyset$ is expanded out we see that all quantifiers in ϕ are bounded[29].

So, for *Lemma B.2 (Box Introduction)*, consider an arbitrary interpreted initial segment $\vec{V}_0$.

By *Lemma M.7* we can have ($\lozenge$) $\vec{V}$ with a successor closed w. And by $\diamond$ Ignoring (*Axiom 8.3*) we can infer the corresponding $\lozenge_{\vec{V}_0}$ claim i.e.,

$$\lozenge_{\vec{V}_0}\left[\vec{\mathcal{V}}(V) \wedge (\exists x \mid \text{set}(x))\phi(x)\right]$$

Enter this $\lozenge_{\vec{V}_0}$ context. By *Lemma K.7 (Hierarchy Extending)* we can have ($\lozenge_{\vec{V}_0,\vec{V}}$) $V_1 \geq V_0$ such that f isomorphicly maps V to an initial segment of $V_{\_} \leq V_1$. Enter this $\diamond_{\vec{V}_0,\vec{V}}$ context and importing all relevant facts. By the *Theorem I.1 (Isomorphism Lemma)* we can infer $(\exists x \mid \text{set}_{\_}(x))\phi[\in/\in_{\_}](x)$. Via the *Theorem L.1 (Translation Lemma)* we can infer $(\exists x \mid \text{set}_1(x))\phi[\in/\in_1](x)$. That is,

$$\vec{\mathcal{V}}(V_0) \wedge V_1 \geq V_0 \wedge (\exists x \mid \text{set}_1(x))\phi[\in/\underset{1}{\in}](x)$$

By *Lemma L.1 (Pointwise Tweaking)* it is possible ($\lozenge_{\vec{V}_0,\vec{V},V_1}$)) that $\vec{V}_1 \geq_x \vec{V}_0$ with $\rho_1(\ulcorner x \urcorner)$ is a successor closed set, i.e., using the notation from *Lemma L.6 (Bounded Quantifiers)*

$$\lozenge_{\vec{V}_0,\vec{V},V_1} \vec{V}_1 \underset{x}{\geq} \vec{V}_0 \wedge \phi^{\vec{V}_1}$$

Enter this $\lozenge_{\vec{V}_0,\vec{V},V_1}$ context. By the *Lemma L.6 (Bounded Quantifiers)* it follows that

$$\vec{V}_1 \underset{x}{\geq} \vec{V}_0 \wedge t_1(\phi)$$

Leaving all $\lozenge$ contexts and applying Diamond Collapsing (*Lemma B.8*) we derive

$$\lozenge_{\vec{V}_0} \vec{V}_1 \underset{x}{\geq} \vec{V}_0 \wedge t_1(\phi)$$

But this is just $t_0((\exists x)\phi)$. Hence, we have $\vdash t_0((\exists x)\phi)$ Applying *Lemma B.2 (Box Introduction)* we infer

$$\Box\vec{\mathcal{V}}(\overrightarrow{V_0}) \to t_0(\exists x)\phi)$$

This completes the proof as this is just the desired conclusion

[29] Note that $\emptyset \in x$ abbreviates $(\exists z \in x)\neg(\exists w \in z))$.

$$t((\exists x)\phi)$$

■

M.10 Replacement

The axiom schema of replacement asserts that the image of a set under any definable function will also fall inside a set.

Proposition M.8 (Potentialist Replacement). *Let θ be any formula in the language of ZFC whose free variables are $x, y, a, w_1, \ldots, w_n$, so that, in particular, b is not free in θ. Then*

$$t(\forall a \forall w_1 \forall w_2 \ldots \forall w_n [\forall x(x \in a \rightarrow \exists!\, y\, \theta) \rightarrow \exists b \forall x(x \in a \rightarrow \exists y(y \in b \wedge \theta))])$$

By *Theorem 9.1 (Logical Closure of Translation)* it's enough to prove that

$$t(\forall a \forall w_1 \forall w_2 \ldots \forall w_n [\forall x(x \in a \rightarrow \exists y\, \theta) \rightarrow \exists b \forall x(x \in a \rightarrow \exists y(y \in b \wedge \theta))])$$

So speaking loosely (in terms of quantifying in), we want to show that following. Given an initial segment V and a set a in V, if for every $x \in a$ it's logically possible that there is some initial segment V_x extending V and a set y_x in V_x making the potentialist translation of $\theta(x, y_x)$ true then it's logically possible to have a single set b in some $V_b \geq V$ that containing all those witnesses. By the *Corollary L.1 (General Translation)* potentialistic truth is absolute (i.e., all extensions of V_x agree on the truth value of the potentialist translation of $\theta(x, y_x)$), unlike the notion of truth in a model. So it is enough to ensure that for each $x \in a$ that V_b extends some $\widehat{V_x}$ containing a $y_x \in b$ satisfying $t(\theta(x, y_x))$.

Proof. By *Theorem 9.1 (Logical Closure of Translation),* it's enough to prove an equivalent (over the remaining axioms of ZF) version of replacement that relaxes the requirement that θ be functional.

$$\vdash t(\forall a \forall w_1 \forall w_2 \ldots \forall w_n [\forall x(x \in a \rightarrow \exists y\, \theta(x, y)) \rightarrow \exists b \forall x(x \in a \rightarrow \exists y(y \in b \wedge \theta(x, y)))])$$

Using *Lemma B.8 (Diamond Collapsing)* and this claim can be simplified to:

$$\Box[\vec{\mathcal{V}}(\vec{V}_{n+1}) \rightarrow [t_{n+1}\big((\forall x)(x \in a \rightarrow \exists y \theta(x, y))\big) \rightarrow t_{n+1}(\exists b \forall x(x \in a \rightarrow \exists y(y \in b \wedge \theta(x, y))))$$

So for *Lemma B.2 (Box Introduction)* we will consider an arbitrary situation in which

$$\vec{\mathcal{V}}(\vec{V}_{n+1}) \wedge t_{n+1}\Big(\big(\forall x\big)\big(x \in a \rightarrow \exists y \theta(x, y)\big)\Big) \qquad (M12)$$

And we will try to show that $t_{n+1}(\exists b \forall x(x \in a \rightarrow \exists y(y \in b \wedge \theta))$ holds in this situation.,i.e.,

$$\Diamond_{\vec{V}_{n+1}}\,(V_{n+2} \underset{b}{\geq} V_{n+1} \wedge t_{n+2}(\forall x(x \in a \rightarrow \exists y(y \in b \wedge \theta)))$$

As noted in Chapter *9,* our strategy will be to use *Axiom 8.13 (Amalgamation)* to establish the possibility of a $\hat{V}$ which specifies, for each x in $\rho_{n+1}(\ulcorner a \urcorner)$, a way, $\vec{V}_x^*$, of assigning 'x' to this x and 'y' to some y which makes the potentialist translation of (θ) true. We will then use the *Theorem K.1 (Hierarchy Combining)* to build V_Σ which contains (isomorphic images of) all these choices for 'y,' and add one layer to it to get $V_{\Sigma+1} = V_{n+2}$ containing a set which has exactly these isomorphic images of choices for 'y' as elements.

In our initial assumption *(M12),* i.e., $\vec{\mathcal{V}}(\vec{V}_{n+1}) \wedge t_{n+1}((\forall x)(x \in a \to \exists y \theta(x,y)))$, writing out the second conjunct formally yields:

$$\Box_{\vec{V}_{n+1}} (\vec{V}_{n+2} \underset{x}{\geq} \vec{V}_{n+1} \wedge \rho_{n+2}(\ulcorner x \urcorner) \underset{n+2}{\in} \rho_{n+2}(\ulcorner a \urcorner) \to \Diamond_{\vec{V}_{n+2}} [\vec{V}_{n+3} \underset{y}{\geq} \vec{V}_{n+2} \wedge \mathrm{t}_{n+3}(\theta)]) \quad (M13)$$

That is: however $\vec{V}_{n+2} \geq_x \vec{V}_{n+1}$ assigns 'x' to a set belonging to $\rho_{n+2}(\ulcorner a \urcorner) = \rho_{n+1}(\ulcorner a \urcorner)$ it is possible ($\Diamond_{\vec{V}_{n+2}}$) that $\vec{V}_{n+3} \geq_{\mathrm{y}} V_{n+2}$ which assigns y in a way to make $t_{n+3}(\theta(x,y))$ true.

M.1 Deploying Amalgamation

I will use *Axiom 8.13 (Amalgamation)* to show that there could ($\Diamond_{\vec{V}_{n+1}}$) be a $\hat{V}$ which puts together witnesses to the above extendability claim *(M13)* .

Specifically there could be a $\hat{\vec{V}}$ which codes up[30], for each object/position $x \in_{n+1} \rho_{n+1}(\ulcorner a \urcorner)$, an initial segment $\vec{V}_x^*$ which:

- assigns 'x' to x
- satisfies $\vec{V}_x^* \geq_{\mathrm{x,y}} \vec{V}_{n+1}$ and hence doesn't tamper with the assignment of any parameters in θ, (i.e., any variables $w_1 \ldots w_m$ other than x and y which are free in θ)
- satisfies $\mathrm{t}^*_{n+3}(\theta)$ (i.e. $t_{n+3}(\theta)[\vec{V}_{n+3}/\vec{V}*]$).

To satisfy the conditions of *Axiom 8.13 (Amalgamation)*, we start by using *Axiom 8.4 (Simple Comprehension)* to establish the possibility that I applies to the $x \in_{n+1} \rho_{n+1}(\ulcorner a \urcorner)$, i.e.,

$$\Diamond_{\vec{V}_{n+1}} (\forall k)(I(k) \leftrightarrow k \underset{n+1}{\in} \rho_{n+1}(\ulcorner a \urcorner)) \qquad (M14)$$

Enter this $\Diamond_{\vec{V}_{n+1}}$ context. To apply *Axiom 8.13 (Amalgamation)* we must establish that, however Q selects a unique object from this index collection I, we can have a corresponding V_x^* with the properties listed above. That is:

$$\Box_{\vec{V}_{n+1},I} [(\exists! x \mid Q(x))(I(x)) \to \Diamond_{\vec{V}_{n+1},Q} \Phi]$$

where Φ expresses the property (in terms of the unique x satisfying Q) that we each V_x should satisfy. In this case that is

$$\Phi = \quad \vec{V}^* \underset{x,y}{\geq} \vec{V}_{n+1} \wedge Q(\rho^*(\ulcorner x \urcorner)) \wedge \mathrm{t}^*_{n+3}(\theta)$$

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[30] That is, it takes x as an extra parameter and for each x the remaining places satisfy the definition of an initial segment.

We will prove this by *Lemma B.2 (Box Introduction)*. So, consider an arbitrary scenario (holding fixed $\vec{V}_{n+1}, I$ facts) in which $(\exists! x \mid Q(x))(I(x))$. As we are holding fixed $\vec{V}_{n+1}$ we may assume that $\vec{V}_{n+1}$ is an interpreted initial segment. So by *Lemma L.1 (Pointwise Tweaking)* we can have ρ_{n+2} assign 'y' to an object satisfying Q. That is:

$$\Diamond_{\vec{V}_{n+1},Q,I}\,[\vec{V}_{n+2} \underset{x}{\geq} \vec{V}_{n+1} \wedge Q(\rho_{n+2}(\ulcorner x \urcorner))]$$

Enter this $\Diamond_{V_{n+1},Q,I}$ context. By $(\exists! x \mid Q(x))(I(x))$ and our characterization of the index property I (the sentence inside equation *(M14)*, we can derive that $\rho_{n+2}(\ulcorner x \urcorner)$ is $x \in_{n+1} \rho_{n+1}(\ulcorner a \urcorner)$. So we have the antecedent of the conditional inside equation *(M13)* namely:

$$\vec{V}_{n+2} \underset{x}{\geq} \vec{V}_{n+1} \wedge \rho_{n+2}(\ulcorner x \urcorner) \underset{n+2}{\in} \rho_{n+2}(\ulcorner a \urcorner)$$

So importing and applying *Lemma B.3 (Box Elimination)* to *(M13)* lets us derive its consequent. Then applying $\Diamond$ Ignoring (*Axiom 8.3*) to add Q, I to the subscript (as the sentence inside the $\Diamond$ above is CR: $\vec{V}_{n+2}, \vec{V}_{n+3}$) gives us

$$\Diamond_{\vec{V}_{n+1},\vec{V}_{n+2},Q,I}\left(\vec{V}_{n+3} \underset{\mathrm{y}}{\geq} \vec{V}_{n+2} \wedge \mathrm{t}_{n+3}(\ulcorner \theta \urcorner)\right)$$

Entering this context and importing the fact that $\vec{V}_{n+2} \geq_x \vec{V}_{n+1}$ and $Q(\rho_{n+2}(\ulcorner x \urcorner))$ and using *Axiom 8.7 (Logical Closure)* we can derive:

$$\Diamond_{\vec{V}_{n+1},\vec{V}_{n+2},Q,I}\,\vec{V}_{n+3} \underset{x,y}{\geq} \vec{V}_{n+1} \wedge Q(\rho_{n+3}(\ulcorner x \urcorner)) \wedge \mathrm{t}_{n+3}(\theta)$$

Now dropping out of the two enclosing $\Diamond$ contexts and applying *Lemma B.8 (Diamond Collapsing)* and *Lemma B.4 (Diamond Reducing)* yields

$$\Diamond_{\vec{V}_{n+1},Q}\,[\vec{V}_{n+3} \underset{x,y}{\geq} \vec{V}_{n+1} \wedge Q(\rho_{n+3}(\ulcorner x \urcorner)) \wedge \mathrm{t}_{n+3}(\theta)]$$

Applying *Axiom 8.5 (Relabeling)* to replace $\vec{V}_{n+3}$ with $\vec{V}^*$ completes our derivation of $\Diamond_{V_{n+1},Q}\,\Phi$ from the assumption that $(\exists! x \mid Q(x))(I(x))$.

Thus we have

$$\vec{\mathcal{V}}(\overrightarrow{V_{n+1}}), \Diamond_{\vec{V}_{n+1}}\,(\forall k)(I(k) \leftrightarrow k \underset{n+1}{\in} \rho_{n+1}(\ulcorner a \urcorner)) \vdash (\exists! x \mid Q(x))(I(x)) \rightarrow \Diamond_{V_{n+1},Q}\,\Phi$$

As both the assumptions used above are content restricted to $\Box_{\vec{V}_{n+1},I}$ we may use *Lemma B.2 (Box Introduction)* to infer

$$\Box_{\vec{V}_{n+1},I}\,[(\exists! x \mid Q(x))(I(x)) \rightarrow \Diamond_{\vec{V}_{n+1},Q}\,\Phi]$$

This completes our proof of the antecedent of *Axiom 8.13 (Amalgamation)*. So by applying *Axiom 8.13 (Amalgamation)* we can infer

$$\Diamond_{\vec{V}_{n+1},I}\big(\quad (\forall y)(\forall x)(\forall x')\big[(\neg x = x' \wedge \pi(y,x) \wedge \pi(y,x') \rightarrow y \in \text{Ext}(\vec{V}_{n+1})\big] \wedge$$
$$\Box_{\vec{V}_{n+1},I,\hat{\vec{V}}}\big[\,(\exists!\, x \mid Q(x))(I(x) \wedge \Psi(x)) \rightarrow \Phi)\big) \qquad (M15)$$

where $\pi(y,x)$ asserts that y appears in some tuple ending with x satisfying some $\hat{\in}, \hat{<}, \hat{@}$ or $\hat{\rho}$ (i.e., informally $\pi(y,x) \leftrightarrow \text{Ext}(\hat{\in}\,(\cdot,x), \hat{<}\,(\cdot,x), \hat{@}(\cdot,x), \hat{\rho}(\cdot,x))(y))$ and $\Psi(x)$ asserts that $\vec{V}^*_{n+3}$ is equal to V_x.

$$\begin{aligned}\Psi(x) = \quad & (\forall z,y)\left(z \underset{n+3}{\overset{*}{\in}} y \leftrightarrow \hat{\in}\,(z,y,x)\right) \wedge \\ & (\forall o,u)\left(o \underset{n+3}{\overset{*}{<}} u \leftrightarrow \hat{<}\,(o,u,x)\right) \wedge \\ & (\forall o,z)\big(@^*_{n+3}(z,o) \leftrightarrow \hat{@}(z,o,x)\big) \wedge \\ & (\forall z,n)(\rho^*_{n+3}(n) = z \leftrightarrow \hat{\rho}(n,x) = z)\end{aligned}$$

Intuitively, this tells us that it's logically possible to have a $\hat{\vec{V}}$ which provides a parameterized witness to the property Φ. Note that the first line of equation *(M15)* asserts that the overlap of V_x and $V_{x'}$ (where these are given by substituting x into the last place of the relations in $\hat{\vec{V}}$) for $x \neq x'$ is contained in $\vec{V}_{n+1}$ while the second line tells us that for any choice of x satisfying I if $V^*_{n+3} = V_x$ then Φ is satisfied by the pair V^*_{n+3} and x.

M.2 *Constructing V_{n+2} and ρ_{n+2}*

M.2.1 Strategy

Now we ultimately need to show that

$$t_{n+1}(\exists b \forall x(x \in a \rightarrow \exists y(y \in b \wedge \theta)))$$

or, equivalently,

$$\Diamond_{\vec{V}_{n+1}}\big(\vec{V}_{n+2} \underset{b}{\geq} \vec{V}_{n+1} \wedge t_{n+2}(\forall x(x \in a \rightarrow \exists y(y \in b \wedge \theta(x,y))))$$

So we want to construct a $\vec{V}_{n+2} \geq_b \vec{V}_{n+1}$ which assigns 'b', so as to make $t_{n+2}(\forall x(x \in a \rightarrow \exists y(y \in b \wedge \theta(x,y)))$ true.

My strategy will be to use *Theorem K.1 (Hierarchy Combining)* to put together the initial segments V_x into a single $V_\Sigma \geq V_{n+1}$ which extends (isomorphic copies of) each V_x and thus contains an image y_x for each x in a under θ. Then by applying *Lemma K.7 (Hierarchy Extending)* we will extend V_Σ to a $V_{\Sigma+1}$ containing a set b, collecting together all the witnesses y_x. $\vec{V}_{n+2}$ will then be the structure $V_{\Sigma+1}$ paired with an assignment of $\ulcorner b \urcorner$ to this set b.

M.2.2 Using the Hierarchy Combining Lemma to get V_Σ

Applying the *Theorem K.1 (Hierarchy Combining)* requires we demonstrate that

$$\Box_{\hat{V},I,\vec{V}_{n+1}}\,(\exists!\, x \mid Q(x))(I(x) \wedge \Upsilon(x)) \rightarrow \vec{\mathcal{V}}(V^*)]$$

where

$$\begin{aligned}\Upsilon(x) = \quad & (\forall z, y)\left(z \overset{*}{\in} y \leftrightarrow \widehat{\in}\,(z, y, x)\right) \wedge \\ & (\forall o, u)\left(o \overset{*}{<} u \leftrightarrow \widehat{<}\,(o, u, x)\right) \wedge \\ & (\forall o, z)\big(@^*(z, o) \leftrightarrow \widehat{@}(z, o, x)\big)\end{aligned}$$

This is very close to the $\Box$ claim we already have in equation *(M15)*. And Φ clearly implies $\vec{\mathcal{V}}(V^*)$, so entering the $\Diamond_{\vec{V}_{n+1},I}$ from equation *(M15)* and then applying *Lemma H.2 (Box Closure)* yields:

$$\Box_{\vec{V}_{n+1},I,\hat{V}}\left[(\exists!\, x \mid Q(x))(I(x) \wedge \Psi(x)) \rightarrow \vec{\mathcal{V}}(V^*))\right]$$

It remains to handle the wrinkle that Ψ differs from Υ in also requiring that ρ^* applies as per $\hat{\rho}_{n+3}$, so the antecedent of the conditional we have is slightly stronger than the antecedent of the conditional we want.

So suppose for *Lemma B.2 (Box Introduction)* that $(\exists!\, x \mid Q(x))(I(x) \wedge \Upsilon(x))$ (while holding fixed the $\vec{\hat{V}}, I, \vec{V}_{n+1}$ facts). Then by *Axiom 8.4 (Simple Comprehension)*:

$$\Diamond_{\vec{V}_{n+1},I,\hat{\vec{V}},Q,V^*}\,(\exists!\, x \mid Q(x))(I(x) \wedge \Upsilon(x)) \wedge (\forall z, n)(\rho^*_{n+3}(n) = z \leftrightarrow \hat{\rho}(n, x) = z) \qquad (M16)$$

Entering this $\Diamond_{\vec{V}_{n+1},I,\hat{\vec{V}},Q,V^*}$ context we can infer $(\exists!\, x \mid Q(x))(I(x) \wedge \Upsilon(x))$ and import equation *(M16)* . Applying *Lemma B.3 (Box Elimination)* to *(M16)*, we deduce $\vec{\mathcal{V}}(V^*)$. And as $\vec{\mathcal{V}}(V^*)$ is content restricted to V^*, so by *Axiom 8.2 (Diamond Elimination)* we can export it from this $\Diamond_{\vec{V}_{n+1},I,\hat{\vec{V}},Q,V^*}$ context.

Thus we've proved

$$(\exists!\, x \mid Q(x))(I(x) \wedge \Psi(x)) \rightarrow \vec{\mathcal{V}}(V^*))$$

using only the assumption that equation *(M16)*, which is content restricted to $\hat{V}, I, \vec{V}_{n+1}$ so we may infer equation *(M15)* using *Lemma B.2 (Box Introduction)*. Applying the *Theorem K.1 (Hierarchy Combining)* to equation *(M15)* lets us infer

$$\begin{aligned}\Diamond_{\vec{V}_{n+1},\hat{V},I}\,[\quad & V_\Sigma \geq V_{n+1} \wedge \underline{v}\mathrm{rng}(f) \subset \mathrm{Ext}(V_\Sigma) \wedge \underline{v} \wedge \\ & \Box_{\hat{V},I,V_\Sigma,f}\,((\exists!\, x \mid Q(x))(I(x) \wedge \Upsilon(x)) \rightarrow \Diamond_{\hat{V},I,f,V_\Sigma}\,(V^* \underset{f}{\cong} V^- \wedge V^- \leq V_\Sigma))]\end{aligned} \qquad \text{(M17)}$$

This asserts the possibility of an initial segment V_Σ which, as discussed above, extends an isomorphic copy of V_x (represented here by V^*) for any x satisfying I.

M.2.3 Forming $V_{n+2}, \rho_{n+2}(b)$

Enter the $\Diamond_{\vec{V}_{n+1},\hat{\vec{V}},I}$ context from equation *(M17)* .

Now by the Proper Extension Lemma *Lemma K.4* we can infer the possibility of V_{n+2} which adds (at least) one layer of sets to V_Σ. And applying $\Diamond$ Ignoring (*Axiom 8.3*) to add subscripts we can conclude

$$\Diamond_{\vec{V}_{n+1},\hat{\vec{V}},I,V_\Sigma,f} V_{n+2} \geq V_\Sigma \wedge (\exists x)(\mathrm{ord}_{n+2}(x) \wedge (\forall y)(\mathrm{ord}_\Sigma(y) \rightarrow y \underset{n+2}{<} x))$$

Entering the above $\Diamond$ context and applying Simple Comprehension (*Axiom 8.4*) we can infer that H could apply to exactly the sets we want to be elements of $\rho_{n+2}(\ulcorner b \urcorner)$, i.e.,

$$\Diamond_{\vec{V}_{n+1},\hat{\vec{V}},I,V_\Sigma,f,V_{n+2}} (\forall y')(H(y') \leftrightarrow (\exists x)(\exists y)(I(x) \wedge \hat{\rho}(\ulcorner y \urcorner, x) = y \wedge f(y) = y'))$$

Note that the biconditional above says $H(y')$ iff there is some x such that $I(x)$ and y' is the image of the $\rho_x^*(\ulcorner y \urcorner))$ selected by $\vec{V}_x^*$.

Now we need to show that V_{n+2} contains a set whose elements are exactly those y' such that $H(y')$. By the fact that $\mathrm{rng}(f) \subset \mathrm{Ext}(V_\Sigma)$ using equation *(M17)* it follows that there is a set b in V_{n+2} whose elements are exactly the y' satisfying $H(y')$. Applying the characterization of I inside equation *(M14)* we can translate the definition of b in terms of I into one in terms of the set a. In particular, we can deduce that b must satisfy

$$(\forall y')[y' \underset{n+2}{\in} b \leftrightarrow (\exists x)(\exists y)(x \underset{n+1}{\in} \rho_{n+1}(\ulcorner a \urcorner) \wedge \hat{\rho}(\ulcorner y \urcorner, x) = y \wedge f(y) = y')$$

Finally applying *Lemma L.1 (Pointwise Tweaking)* to this existential claim and V_{n+2}, ρ_{n+1} (importing the fact that $V_{n+2} \geq V_\Sigma \geq V_{n+1}$) lets us conclude that

$$\vec{V}_{n+2} \underset{b}{\geq} \vec{V}_{n+1} (\forall y')[y' \underset{n+2}{\in} \rho_{n+2}(\ulcorner b \urcorner) \leftrightarrow$$
$$(\exists x)(\exists y)(x \underset{n+1}{\in} \rho_{n+2}(\ulcorner a \urcorner) \wedge \hat{\rho}(\ulcorner y \urcorner, x) = y \wedge f(y) = y')$$

We now apply *Lemma H.6 (Diamond Simplification)* to collapse the $\Diamond$ contexts we entered in the last two subsections and importing the interior of equation *(M14)* to conclude

$$\begin{aligned}
\Diamond_{\vec{V}_{n+1},\hat{\vec{V}},I} \quad & (\forall k)(I(k) \leftrightarrow k \underset{n+1}{\in} \rho_{n+1}(\ulcorner a \urcorner)) \\
& (\forall y)(\forall x)(\forall x')\big[(\neg x = x' \wedge \pi(y,x) \wedge \pi(y,x') \rightarrow y \in \mathrm{Ext}(\vec{V}_{n+1})\big] \wedge \\
& \Box_{\vec{V}_{n+1},I,\hat{\vec{V}}} [(\exists! x \mid Q(x))(I(x) \wedge \Psi(x)) \rightarrow \Phi) \\
[\quad & V_\Sigma \geq V_{n+1} \wedge \underline{v}\mathrm{rng}(f) \subset \mathrm{Ext}(V_\Sigma) \wedge \underline{v} \\
& \vec{V}_{n+2} \underset{b}{\geq} \vec{V}_{n+1} \wedge (\forall y')[y' \underset{n+2}{\in} \rho_{n+2}(\ulcorner b \urcorner) \leftrightarrow (\exists x)(\exists y)(x \underset{n+1}{\in} \rho_{n+2}(\ulcorner a \urcorner) \wedge \\
& \hat{\rho}(\ulcorner y \urcorner, x) = y \wedge f(y) = y') \wedge \\
& \Box_{\hat{\vec{V}},I,V_\Sigma,f} ((\exists! x \mid Q(x))(I(x) \wedge \Upsilon(x)) \rightarrow \Diamond_{\hat{V},I,f,V_\Sigma} (V^* \underset{f}{\cong} V^- \wedge V^- \leq V_\Sigma))]
\end{aligned} \tag{M17}$$

M.3 *Verification for $\vec{V}_{n+2}$*

It remains only to show that the $\vec{V}_{n+2}$ we have constructed has the properties claimed by the (translation of) the consequent of the replacement axiom schema. Entering the $\Diamond_{\vec{V}_{n+1}}$ context above we need to show

$$\vec{V}_{n+2} \underset{b}{\geq} \vec{V}_{n+1} \wedge t_{n+2}(\forall x(x \in a \rightarrow \exists y(y \in b \wedge \theta))$$

We already know $\vec{V}_{n+2} \geq_b \vec{V}_{n+1}$, and expanding out the second conjunct yields the following.

$$\begin{aligned} \Box_{\vec{V}_{n+2}} [\quad & V_{n+3} \underset{x}{\geq} V_{n+2} \wedge \rho_{n+3}(\ulcorner x \urcorner) \underset{n+3}{\in} \rho_{n+3}(\ulcorner a \urcorner) \rightarrow \\ & \Diamond_{\vec{V}_{n+3}} [V_{n+4} \underset{\mathrm{y}}{\geq} V_{n+3} \wedge \rho_{n+4}(\ulcorner y \urcorner) \underset{n+4}{\in} \rho_{n+4}(\ulcorner b \urcorner) \wedge t_{n+4}(\theta)] \end{aligned} \qquad (M18)$$

We first note that, as the sentence inside the $\Box_{\vec{V}_{n+2}}$ in equation *(M18)* is content restricted to $\vec{V}_{n+3}, \vec{V}_{n+2}$. By *Lemma H.1 (Box Ignoring)* it's sufficient to prove (where $\mathcal{L} = \{\vec{V}_{n+2}, \vec{V}_{n+1}, V_\Sigma, \vec{V}_{n+2}, I, \hat{\vec{V}}, f\}$

$$\begin{aligned} \Box_{\mathcal{L}} [\quad & V_{n+3} \underset{x}{\geq} V_{n+2} \wedge \rho_{n+3}(\ulcorner x \urcorner) \underset{n+3}{\in} \rho_{n+3}(\ulcorner a \urcorner) \rightarrow \\ & \Diamond_{\vec{V}_{n+3}} [V_{n+4} \underset{\mathrm{y}}{\geq} V_{n+3} \wedge \rho_{n+4}(\ulcorner y \urcorner) \underset{n+4}{\in} \rho_{n+4}(\ulcorner b \urcorner) \wedge t_{n+4}(\theta)] \end{aligned} \qquad (M19)$$

We will prove this claim by *Lemma B.2 (Box Introduction)*. As the interior of equation *(M16)* is content restricted to $\vec{V}_{n+2}, \vec{V}_{n+1}, V_\Sigma, \vec{V}_{n+2}, I, \hat{\vec{V}}, f$ we may consider an arbitrary scenario in which this holds as well as

$$V_{n+3} \underset{x}{\geq} V_{n+2} \wedge \rho_{n+3}(\ulcorner x \urcorner) \underset{n+3}{\in} \rho_{n+3}(\ulcorner a \urcorner). \qquad (M20)$$

and derive that

$$\Diamond_{\vec{V}_{n+3}} [V_{n+4} \underset{\mathrm{y}}{\geq} V_{n+3} \wedge \rho_{n+4}(\ulcorner y \urcorner) \underset{n+4}{\in} \rho_{n+4}(\ulcorner b \urcorner) \wedge t_{n+4}(\theta)]$$

M.4 *Constructing $\vec{V}_{n+4}$*

We proceed by building a $\vec{V}^*$ equal to $\vec{V}_x$ where $x = \rho_{n+3}(\ulcorner x \urcorner)$. We will then infer that $t^*_{n+3}(\theta)$ holds with respect to this $\vec{V}^*$. We will then define ρ^- to be the isomorphic image of ρ^* under f and use the isomorphism lemma to infer that $t^-_{n+3}(\theta) \overset{\text{def}}{=} t^-_{n+3}(\theta)[V^*/V^-]$ holds with respect to $\overrightarrow{V^-}$ where $V^- \leq V_\Sigma \leq V_{n+3}$. We will then define V_{n+4} to be equal to V_{n+3} and use *Lemma L.1 (Pointwise Tweaking)* to let ρ_{n+4} be equal to ρ_{n+3} excepting only $\rho_{n+4}(\ulcorner y \urcorner)$ which we instead define to be $\rho^-(\ulcorner y \urcorner)$. Then using *Theorem L.1 (Translation)* we infer $t_{n+4}(\theta)$ Note that in what follows we ensure that every $\Diamond$ context we enter we subscript $\mathcal{L}, \vec{V}_{n+3}$ allowing us to import the interior of equation *(M16)* in each context we enter.

As indicated, we start by invoking Simple Comprehension (*Axiom 8.4*) (via *Lemma H.7 (Multiple Definitions)*) to define $\vec{V}^*$ to be equal to $\vec{V}_x$ (i.e., to make $\Psi(x)$ hold from equation *(M15)*) and to define Q to hold of the unique value $\rho_{n+3}(\ulcorner x \urcorner)$. More formally, we deduce the logical possibility ($\Diamond_{\mathcal{L}, \vec{V}_{n+3}}$) of

$$(\forall k)\big(Q(k) \leftrightarrow k = \rho_{n+3}(\ulcorner x \urcorner)\big) \wedge \Psi\big(\rho_{n+3}(\ulcorner x \urcorner)\big) \qquad (M21)$$

and

$$\begin{aligned}&(\forall z, y)\left(z \underset{n+3}{\overset{*}{\in}} y \leftrightarrow (\exists x \mid Q(x))\ \widehat{\in}\ (z, y, x)\right) \wedge \\ &(\forall o, u)\left(o \underset{n+3}{\overset{*}{<}} u \leftrightarrow (\exists x \mid Q(x))\ \widehat{<}\ (o, u, x)\right) \wedge \\ &(\forall o, z)\left(@^*_{n+3}(z, o) \leftrightarrow (\exists x \mid Q(x))\widehat{@}(z, o, x)\right) \wedge \\ &(\forall z, n)(\rho^*_{n+3}(n) = z \leftrightarrow (\exists x \mid Q(x))\hat{\rho}(n, x) = z)\end{aligned} \qquad (M22)$$

Enter this $\Diamond_{\mathcal{L},\vec{V}_{n+3}}$ context. We note for latter use that $\rho^*(\ulcorner\ y\ \urcorner) = \hat{\rho}(\ulcorner\ y\ \urcorner, \rho_{n+3}(\ulcorner\ x\ \urcorner))$

By assumption we have $\rho_{n+3}(\ulcorner\ x\ \urcorner) \in_{n+3} \rho_{n+3}(\ulcorner\ a\ \urcorner)$ and as $\vec{V}_{n+3} \geq_x \vec{V}_{n+2} \geq_b \vec{V}_{n+1}$ we can infer $\rho_{n+3}(\ulcorner\ x\ \urcorner) \in_{n+1} \rho_{n+1}(\ulcorner\ a\ \urcorner)$. Using the charachterization of I we can conclude $I(\rho_{n+3}(\ulcorner\ x\ \urcorner))$. Hence, (as $\Psi(x)$ is implied by equation *(M22)*) we can infer $(\exists!\, x \mid Q(x))(I(x) \wedge \Psi(x))$. Now using equation *(M15)* (as included in equation *(M16)*) we have

$$\Box_{\vec{V}_{n+1}, I, \hat{\vec{V}}}\left[(\exists!\, x \mid Q(x))(I(x) \wedge \Psi(x)) \rightarrow \Phi)\right]$$

Applying *Lemma B.3 (Box Elimination)* lets us deduce

$$\Phi = \vec{V}^* \underset{x,y}{\geq} \vec{V}_{n+1} \wedge Q\left(\rho^*(\ulcorner\ x\ \urcorner)\right) \wedge \mathrm{t}^*_{n+3}\left(\theta(x, y)\right). \qquad (M23)$$

We now move to transfer the fact that $\mathrm{t}^*_{n+3}(\theta(x, y)$ holds to infer that $\mathrm{t}^-_{n+3}(\theta(x, y)$ holds for some $V^- \leq V_\Sigma$. We do this by applying the last conjunct in equation *(M16)*. That is

$$\Box_{\hat{V}, I, V_\Sigma, f}\left((\exists!\, x \mid Q(x))(I(x) \wedge \Upsilon(x)) \rightarrow \Diamond_{\hat{V}, I, f, V_\Sigma}\left(V^* \underset{f}{\cong} V^- \wedge V^- \leq V_\Sigma\right)\right)$$

Applying *Lemma B.3 (Box Elimination)* and $\Diamond$ Ignoring (*Axiom 8.3*) we can infer

$$\Diamond_{\mathcal{L},\vec{V}_{n+3}}\left(V^* \underset{f}{\cong} V^- \wedge V^- \leq V_\Sigma\right) \qquad (M24)$$

Enter this $\Diamond_{\mathcal{L},\vec{V}_{n+3}}$ context and import all the facts we've established so far. Now we specify ρ^- so that $\vec{V}^* \cong_f \vec{V}^-$ so we can apply the *Theorem I.1 (Isomorphism Lemma)*. By *Axiom 8.4 (Simple Comprehension)* we infer

$$\Diamond_{\mathcal{L},\vec{V}_{n+3},Q,\vec{V}_{n+4},V^-}\left((\forall x)(\forall y)[\rho^-(x) = y \leftrightarrow y = f(\rho^*(x))]\right) \qquad (M25)$$

Entering this context and importing all necessary facts we can derive that $\vec{V}^* \cong_f \vec{V}^-$ from equation *(M24)* and equation *(M25)* . Thus, from $t^*_{n+3}(\theta)$ we can apply the *Theorem I.1 (Isomorphism Lemma)* to deduce $t^-_{n+3}(\theta)$ (where $t^-_{n+3}(\theta) \overset{\text{def}}{=} t^-_{n+3}(\theta)[V_{n+3}/V^-]$).

We now use Simple Comprehension (*Axiom 8.4*) (via *Lemma H.7 (Multiple Definitions Lemma)*) to define V_{n+4} to be equal to V_{n+3} and, entering this context, use *Lemma L.1 (Pointwise Tweaking)* to let ρ_{n+4} be equal to ρ_{n+3} excepting only $\rho_{n+4}(\ulcorner\ y\ \urcorner)$ which we instead define to be $\rho^-(\ulcorner\ y\ \urcorner)$. Using $\Diamond$ Ignoring (*Axiom 8.3*) we can expand the subscript of the $\Diamond$ introduced by *Lemma L.1 (Pointwise Tweaking)* to be $\mathcal{L}, \vec{V}_{n+3}, Q, \vec{V}_{n+4}, V^-$ and enter this context importing all the (suitably content restricted) facts derived so far.

We now argue that

$$\vec{V}_{n+4} \underset{\mathrm{y}}{\geq} \vec{V}_{n+3} \wedge \rho_{n+4}(\ulcorner y \urcorner) \underset{n+4}{\in} \rho_{n+4}(\ulcorner b \urcorner) \wedge t_{n+4}(\theta). \quad (M26)$$

We already have that $\vec{V}_{n+4} \geq_{\mathrm{y}} \vec{V}_{n+3}$ and by *(M20)* we have $\rho_{n+3}(\ulcorner y \urcorner) \in_{n+3} \rho_{n+3}(\ulcorner b \urcorner)$ and thus as $\vec{V}_{n+4} \geq_{\mathrm{y}} \vec{V}_{n+3}$ we have $\rho_{n+4}(\ulcorner y \urcorner) \in_{n+4} \rho_{n+4}(\ulcorner b \urcorner)$ so we now work to show that $t_{n+4}(\theta)$. To this end we establish that ρ^{-} and ρ_{n+4} agree on all variables free in θ so that we may apply the *Theorem L.1 (Translation)* to go from $t^{-}_{n+3}(\theta)$ to ρ_{n+4} and ρ^{*} $t_{n+4}(\theta)$.

To this end we note that ρ_{n+4} and ρ^{*} agree with ρ_{n+1}, and hence each other, on all variables other than x,y and b since

$$\vec{V}_{n+4} \underset{y}{\geq} \vec{V}_{n+3} \underset{x}{\geq} \vec{V}_{n+2} \underset{b}{\geq} \vec{V}_{n+1}$$

and

$$\vec{V}^{*} \underset{x,y}{\geq} \vec{V}_{n+1}$$

And by equation *(M21)* and equation *(M23)* we can conclude that ρ_{n+3} (and hence ρ_{n+4}) and ρ^{*} agree on x as well. Furthermore, by equation *(M20)* we know that $\rho_{n+3}(\ulcorner x \urcorner) \in_{n+3} \rho_{n+3}(\ulcorner a \urcorner)$ and as $\rho_{n+3}(\ulcorner a \urcorner) = \rho_{n+1}(\ulcorner a \urcorner)$ it follows that $\rho^{*}(\ulcorner x \urcorner)$ is in V_{n+1} as well. Since $V_{n+1} \leq V^{*}$ and $V_{n+1} \leq V^{-}$ by *Lemma K.1 and K.2 (Isomorphism Agreement Lemmas)* it follows that f is the identity on V_{n+1} and thus ρ^{-} agrees with ρ^{*} and hence ρ_{n+4} on all variables other than y and b.

But by assumption b isn't free in θ and our application of the *Lemma L.1 (Pointwise Tweaking)* defined $\rho_{n+4}(\ulcorner y \urcorner)$ to be and $\rho^{-}(\ulcorner y \urcorner)$ so we may apply *Theorem L.1 (Translation)* (see notes after the theorem regarding substituting V^{*} for V_{n+3}) to infer $t_{n+4}(\theta)$ from $t^{-}_{n+3}(\theta)$. This completes our proof that equation *(M26)*. Leaving the most recent $\lozenge$ context and applying *Lemma H.6 (Diamond Simplification)* yields

$$\lozenge_{\vec{V}_{n+3}} (\overrightarrow{V_{n+4}} \underset{\mathrm{y}}{\geq} \overrightarrow{V_{n+3}} \wedge \rho_{n+4}(\ulcorner y \urcorner) \underset{n+4}{\in} \rho_{n+4}(b) \wedge t_{n+4}(\theta))$$

As this fact is content restricted to $\vec{V}_{n+3}$ we can pull it out of all intermediate $\lozenge$ contexts (because they all subscript $\vec{V}_{n+3}$) to establish that *(M26)* as desired. Since our proof of *(M26)* relied only on the assumption of the antecedent

$$V_{n+3} \underset{x}{\geq} V_{n+2} \wedge \rho_{n+3}(\ulcorner x \urcorner) \underset{n+3}{\in} \rho_{n+3}(\ulcorner a \urcorner)$$

and facts content restricted to $\mathcal{L}$ we can apply *Lemma B.2 (Box Introduction)* to infer equation *(M.19)* which, by the discussion at the top of subsection M.10.2 suffices to complete the proof. ■