PHYSICAL POSSIBILITY AND DETERMINATE NUMBER THEORY

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ABSTRACT. It's currently fashionable to take Putnamian model theoretic worries seriously for mathematics, but not for discussions of ordinary physical objects and the sciences. But I will argue that (under certain mild assumptions) merely securing determinate reference to *physical* possibility suffices to rule out non-standard models of our talk of numbers. So anyone who accepts realist reference to physical possibility should not reject reference to the standard model of the natural numbers on Putnamian model theoretic grounds.

1. INTRODUCTION

In [14] and [13] Putnam uses the possibility of unintended mathematical models for our best scientific/physical theories to raise a challenge for realist reference to mind-independent objects, and realist claims to have a categorical conception of the structure of the natural numbers. It's currently fashionable to take such model theoretic worries seriously for mathematics, but not for discussions of ordinary physical objects and the sciences[1].

[1]Admittedly this combination of attitudes is more often held implicitly and acknowledged under questioning than explicitly defended in print. People are willing to advance Putnamian model theoretic arguments as concerns for mathematical reference while feeling free to ignore them as far as scientific or modal vocabulary is concerned. But even if fashions in my circle aren't general, I think the combination of views is interesting and tempting enough to be worth refuting.

In this paper I will attack this combination of views. I will argue that (under certain mild physical assumptions) merely securing determinate reference for certain *physical* vocabulary suffices to rule out nonstandard models of our talk of numbers. So philosophers who think we can (somehow) determinately refer to physical possibility should not reject determinate reference to natural numbers on Putnamian model-theoretic grounds. Note that I'm not arguing that the combination of physical determinacy and number theoretic indeterminacy is itself incoherent, but rather that if we can secure determinate reference for physical vocabulary then this suffices to block one specific reason accepting number theoretic indeterminacy (Putnam's model theoretic skepticism).

In section 2 I will set up the model-theoretic challenge to mathematical/physical realism I want to consider. In section 3 I'll review some limitations and problems for previous work by Hartry Field and Vann McGee discussing how connections between mathematical vocabulary and physical vocabulary can rule out nonstandard models. In section 4 I will spell out my main argument and in section 5 I will discuss some helpful variants on this argument.

2. Putnam's Model-Theoretic Challenge(s)

2.1. **A Challenge for Physical and Mathematical Realists.** Let's begin by considering Putnam's model-theoretic challenge, as it applies to the natural numbers.

From a naive realist point of view, it appears that all number theoretic statements have a determinate truth value, and that we can refer to the natural numbers (at least up to isomorphism). A natural place to look for an explanation of our ability to refer to the natural numbers is our beliefs about them. So one might hope to appeal to our acceptance of various statements of first order number theory (plus our grasp of various first order logical connectives) to explain how our number talk secures definite reference.

However, by the Lowenhiem-Skolem theorem, any first order theory which has an infinite model (as our first order number theory must), has models of different sizes. Thus our mere beliefs about number theory and grasp of first order logical vocabulary cannot pin down reference to the natural numbers (even up to isomorphism). For example, the standard first order Peano axioms of arithmetic (PA) plausibly articulate part of our concept of numbers. But, these axioms can also be satisfied by non-standard models.

In view of the existence of such non-standard models, one can ask (as Putnam does) the following question. Do we really have a definite concept of 'the structure of the natural numbers' which is not satisfied by any non-standard models? What can such a concept consist of? What is it about us which (perhaps together with other kinds of facts about the world) lets our words like "number" and "plus" take on meanings which rule out such non-standard models? For reasons I won't discuss here, Putnam takes our ability to give standard meanings to first order logical vocabulary for granted in this challenge. I will follow him in doing so.

Accordingly, we can dramatize Putnam's challenge as follows. Imagine some all-knowing interpreter who is dedicated to interpreting our talk about the natural numbers in some unintended fashion while preserving the meaning of our first order logical vocabulary.

Can we cite plausible constraints which our perverse interpreter must honor which prevent her from giving an unintended interpretation our word 'number'? Note that, by the Löwenheim-Skolem theorem mentioned above, merely requiring she interpret first order logical vocabulary normally and make some set of axioms about the natural numbers come out true (e.g., axioms extending PA or embedding the numbers in a larger structure) couldn't provide such a constraint.

If we can give no satisfying answer to Putnam's challenge then, perhaps, we must give up the realist intuition that we can refer to the standard model of the natural numbers. For example, we might allow that our conception of the structure of the natural numbers is vague and allows for a range of acceptable precisifications (corresponding to different non-isomorphic structures satisfying the Peano Axioms), analogous to the range of different acceptable personifications of 'bald' and 'heap'[2].

Failure to answer Putnam's challenge also raises a problem for the common presumption that all statements of arithmetic (even ones we can't decide) have definite truth-values. For one thing, the most common way of accounting for such definite truth values is through definite reference to the natural numbers. But, more directly, combining

[2]C.f. [6]

Gödel's Completeness and Incompleteness theorems [7] tells us that any computably axiomatizable theory extending basic arithmetic - such as our first order number-theoretic beliefs - will have models which disagree on the truth of some number theoretic claim. So, if we concede to the Putnamian skeptic that every model of our first order mathematical beliefs is an equally acceptable precisification of our number concept, then we are forced to conclude that the truth-value of some number theoretic claims is vague or indeterminate.

Of course, if one took our ability to mean full second order quantification for granted, one could use our acceptance of *second order* Peano Arithmetic, (the version of Peano Arithmetic which replaces the first order induction axiom schema with the second order induction axiom below) to explain what is wrong with non-standard models of the natural numbers.

Induction Axiom: $(\forall X)\,[(X(0) \wedge (\forall n)(X(n) \rightarrow X(S(n)))) \rightarrow (\forall m)X(m)]$

For, the non-standard models of PA guaranteed by Löwenheim-Skolem and Completeness arguments (discussed above) don't satisfy this induction axiom, if we take second order quantification to range over *all* possible subsets. However our ability to pin down standard meanings for second order quantifiers is (prima facie) no less mysterious than our ability to latch on to the standard model of the natural numbers. So this fact is not enough to banish Putnamian skeptical worries on its own.

Putnam supplements this mathematics-specific worry with a more general model theoretic challenge to realist approaches to truth and reference. Very crudely, the worry goes like this. From a naive realist point of view, it seems that we can talk about physical objects and grasp scientific concepts in a way that makes it possible for an ideal scientific theory[3] to be wrong. However, any consistent first order theory can be interpreted as speaking truly about (some of) the sets. So why doesn't our ideal scientific theory count as speaking truly of this model? Why isn't it more charitable to interpret this theory as speaking truly of the sets rather than falsely of electrons and rabbits (the apparent subject matter of the theory)?

As noted above, many philosophers are inclined to take model theoretic worries about grasping mathematical concepts seriously, while rejecting this more general challenge. Perhaps this difference can be partly motivated by the fact one can invoke causal contact with objects like rabbits and electrons in answering Putnam's general challenge. But, since mathematical objects are generally taken to be causally inert, we cannot do the same when answering Putnam's challenge with respect to the natural numbers.

I will argue against this combination of views by showing that, given certain mild physical assumptions, merely (somehow) securing definite reference for our physical vocabulary and talk of physical (or metaphysical) possibility would suffice to ensure definite reference to the natural numbers as well.

[3]That is, a theory which would be accepted in the 'ideal limit' of scientific investigation.

2.2. **Clarifying the Challenges.** But first let me make a quick remark about how I am understanding the Putnamian challenges introduced above and what it would take for the realist to answer them.

Like most philosophically interesting skeptical arguments, the Putnamian challenge I'm concerned with in this paper doesn't just highlight the possibility of doubting some realist doctrine. Instead it seems to provide positive reason for doubt, arising from premises which the realist accepts, by revealing a tension *within* the realists' own total philosophical view.

Specifically, the Putnamian skeptic doesn't just doubt that we can determinately refer to mathematical objects/physical objects/causes and ask to be convinced. Rather they attempt to show that, even taking the realist's own theory for granted, one can't explain why we don't refer to some non-standard model.

Accordingly, a realist answering this skeptical challenge may use their whole physical, mathematical and metasemantic theory to explain what facts let their words secure definite realist reference. They may (non-question-beggingly) employ terms like 'set', 'standard model' etc. (and take these to refer unproblematically) in the meta-language when giving this explanation. But, of course, if no such explanation can be given, the Putnamian challenge succeeds – giving us reason to doubt our ability to determinately refer to these objects.

To help keep this dialectical situation clear, we can flesh out the conceit of the perverse interpreter (suggested above) as follows[4].

[4] I claim no originality for this extended conceit, which I take to be something like philosophical folklore.

As before, we imagine the Putnamian skeptical challenge as a game played between the realist and a perverse interpreter who attempts to illustrate the inadequacy of the realist's theory of reference. But now we will suppose that the perverse interpreter is dedicated to offering an unintended interpretation of some third party's speech, while the realist tries to force the interpreter to give the intended interpretation. Call this third party the speaker. The realist wins, i.e., succeeds in blocking the skeptical challenge[5], if they can block - in a principled way - each of the perverse interpreter's unintended interpretations.

In doing this, the realist is permitted to use ordinary set theory and model theory in the metalanguage, to reason about what interpretations are possible (as the skeptic does in setting up the model theoretic challenge). *They* are allowed to talk about ω sequences and what standard vs. nonstandard interpretations of the speaker's number theoretic talk exist (and use set theory to reason about these interpretations). But they cannot presume that *the third person whose definite reference is to be explained* has any such capacity[6].

[5]This is not to say that they succeed in rationally compelling the skeptic to become a realist.

[6]This point (that there is no problem about freely referring to standard models of the natural numbers in the metalanguage) bears some resemblance to standard responses to (one interpretation of) Puntam's infamous and debated 'just more theory' argument.

In [13] Putnam supplements the core model theoretic challenge sketched above with an argument that we cannot appeal to causal theories of reference to rule out unintended interpretations of our mathematical/scientific vocabulary. You might try to say that your word 'rabbit' is better interpreted as referring to rabbits than sets, because you have causal contact with rabbits and not sets. But (Putnam seems to suggest) to say this is just to point out that you accept a certain further theory (apparently about the reference of your own words). And this expanded theory can still be satisfied by a model that takes all of your terms to apply only to sets.

I will be arguing that (under certain physical assumptions) people who take the above skeptical model theoretic challenge to be answerable in the physical case should not take it to be unsolvable in the case of number theory.

Accordingly I will imagine a scenario where the realist has already (somehow!) succeeded in blocking all nonstandard interpretations of the speaker's physical vocabulary. I will consider whether they can then rule out the remaining perverse interpretations which associate their number talk with some nonstandard model, while interpreting all their (mathematics free) physical vocabulary standardly.

Specifically, I will identify a certain sentence which is in some sense definitional for the natural numbers, i.e., we can reasonably say that part of what we mean by the natural numbers is that they make this sentence true. Then, I will argue in the meta-language (presuming mathematical realism in the meta-language just as Putnam does in setting up the problem) that this sentence can *only* be satisfied (given the interpratational rules above) by the standard model of the natural numbers. In making this argument I will presume that a certain

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I follow Lewis [10] Devitt[4] and many others in rejecting this argument (if it is read as a development of the skeptical challenge articulated above) for the following reason. It seems to mistake the proposal that a person's utterances of "rabbit" refer to rabbits *because these utterances have a certain causal relationship to rabbits* and not sets with the claim that they so refer *because the person being interpreted accepts a bunch of sentences* (which appear to articulate a causal theory of reference).
But I won't attempt to add to this debate here. I simply want to note that even if Putnam's supplementary 'just more theory' argument fails (so causal facts can in principle be relevant to explaining reference), we are left with an interesting and troubling skeptical challenge, which has been very philosophically influential. For example David Lewis, while rejecting the 'just more theory' argument takes Putnam's core model theoretic scepticism seriously enough that he lists answering it as an important motivation for his theory of natural kinds[9].
That core challenge is my intended topic in this paper.

plausible principle IRS holds regarding physical possibility. Note that, as IRS is only used as part of the meta-language argument regarding what interpretations obey the rules, we may unproblematically assume mathematical realism in formulating it.

Now let's turn to that argument.

3. Contrast With Field and McGee

My proposal takes inspiration from a pair of existing explanations of determinate arithmetical reference suggested by Hartry Field and Van McGee. Field and McGee each attempt to use our definite reference to certain non-mathematical vocabulary to pin down a definite interpretation (up to isomorphism) for our natural number talk.

One can think of my proposal as attempting to solve problems for Field and McGee by combining elements from each story. However, unlike Field and McGee, I'll only be arguing for the dialectically important conditional that (under certain mild assumptions) '*if* we can somehow secure definite reference to certain physical/metaphysical notions, this suffices rule out nonstandard interpretations of number talk as well'.

I won't suggest that definite physical reference plays a role in the only (or best) explanation for definite mathematical reference.

3.1. **The Language Expansion Approach.** In [11] McGee offers an account of definite reference to the natural numbers which centers on what he calls 'openendedness'[7]. Openendedness is the idea that

[7] See [12] for a related proposal.

we expect all instances of the first order induction axiom schema to continue holding true in any 'logically permissible' extension of our language.

McGee argues (roughly[8]) as follows. Part of our current mathematical practice is the above expectation that the induction schema will remain true in all 'logically permissible' expansions of our language. McGee suggests that this fact helps rule out non-standard models as follows. Suppose (for contradiction) that some nonstandard model M provided an acceptable interpretation of our terms 'natural number', 'successor' etc. Then there could (in some sense) be a god who is able to point to the non-standard model and introduce a term 'smee' which applies counter-inductively to it (i.e., smee applies to 0, and smee$(n) \rightarrow$ smee$(S(n))$, but smee doesn't apply to every 'natural number'). If we met such a god then we could (logic-preservingly) extend our language by adding the term 'smee' from their language to ours. In such a case, we would still expect the induction axiom to hold for formulas involving the term 'smee' which we got from talking to this god. Therefore, interpreting us to mean a nonstandard model is unacceptable, because it would fail to satisfy induction in some extended language.

This proposal faces a number of worries and objections. First, it's prima facie unclear that it's metaphysically possible for a god to introduce a term like 'smee'. For instance, it's not clear how the god could

[8]McGee's actual proposal is somewhat more complicated, in ways that I claim don't effect any of the criticisms discussed here. See pgs 56-68 of [11] for the details I've elided.

refer sufficiently definitely to some proper initial segment of our non-standard model. What can the god do to secure reference in a way we cannot? Are we to imagine a metaphysically impossible scenario where they fly into the realm of abstract objects and point one by one to each of the infinitely many elements in the initial segment? Perhaps this worry is close to what Field has in mind when he writes, 'why can't we just say that we secure definite reference by whatever we are imagining the god to do to secure her reference?' in a criticism of McGee [6].

But McGee tries to head off all such concerns by appealing to the idea that we are committed to the first order induction schema being true in all *logically* possible extensions of our language (whether these are metaphysically possible or not). He writes:

> To say what individuals and classes of individuals the rules of our language permit us to name is easy: we are permitted to name anything at all. For any collection of individuals K there is a logically possible world - though perhaps not a theologically possible world - in which our practices in using English are just what they are in the actual world and in which K is the extension of the open sentence 'x is blessed by God'. So the rules of our language permit the language to contain an open sentence whose extension is K[11].

But, this insouciance comes with a cost. For appealing to a speaker's intention that some principle hold under 'all logically possible extensions of our language' to explain how we have a determinate conception of the natural numbers seems question begging in the current context. We wouldn't accept an answer to Putnam's worries that *just presumed* the speaker being interpreted had a determinate conception of second order quantification. And it's not clear that assuming the speaker can determinately refer to all logically possible linguistic extensions is materially different. If they can somehow intend that the induction schema remain true in all logically permissible expansions of our language (in the above sense which includes languages corresponding to all possible ways of choosing a subset of individuals for a predicate to apply to), why can't they use the same faculty to directly intend that our second order quantifiers range over every possible subset?[9]

One might also worry about whether we can rigorously cash out McGee's notion of 'logically permissible' extensions of our language [10].

My argument will avoid both of these problems. For it makes no claims about 'all logically permissible expansions of our language', so it's not on the hook for clarifying this concept. And it won't presuppose that speakers can refer to and have intentions about 'all logically possible scenarios' involving gods dubbing things (or any metaphysically impossible scenarios at all).

[9]See [2],[15][5].

[10]Sometimes, rather than talking about all logically permissible expansions of contemporary English, McGee talks about all expansions of English which are permitted by the rules of current English noting that, "the rules of English certainly permit the adjunction of additional vocabulary; indeed, we add new singular terms to the language whenever we baptize an infant or bring a new puppy home from the pound, and we add new general terms whenever we devise a new theory, discover a new species, or introduce a new product line."[11]

3.2. **Appeal to the Structure of Time.** In [5] Hartry Field (rather ambivalently) proposes a different way of using physical reference to pin down reference to the (intended structure of the) natural numbers. In effect, he argues that *if* evenly spaced points in time (starting from any point) form a genuine ω sequence[11] (i.e., time has infinite duration and there are only a finite number of seconds between any two times) then this belief can be used to rule out nonstandard interpretations of our number talk.

However Field's story has some limitations and also raises some concerns, which my account will avoid. First, Field's story only pins down definite mathematical reference for people who happen to truly **believe** that time has a certain structure. It doesn't explain how people who keep an open mind about this matter of cosmology could refer to the natural numbers. In contrast, my story won't require that the people whose reference is to be explained have any *beliefs* about contingent physics. It will only require they accept that (a certain instance of) the induction schema for the natural numbers holds with physical necessity) plus some analytic principles about counting listed below. Thus, it will let us defend realist intuitions that even the cosmologically open minded can definitely refer to the natural numbers.

[11] By an ω sequence refers to a plurality of elements which (when considered under some relation $<$), has the same structure as the intended model of the natural numbers, i.e., is comprised of a first element, the successor of that element and so forth. Note that the claim time forms an ω sequence (assuming it is linearly ordered) is equivalent to the claim that if we start marking off one second intervals at any point those marks form an ω sequence.

Second, Field's story requires the assumption that time actually has a certain structure (mirroring that of the intended model of the numbers, as specified above) [12]. In contrast, my argument won't require that the actual laws of physics privilege the standard model of the numbers. Instead it will merely require that a certain kind of infinite sequence of random events is *physically possible*. And in section 5, I will discuss some ways in which even this contingent physical assumption can be dropped.

Thirdly, it's not clear that the mere fact that one of the speaker's scientific hypotheses will come out false if we interpret their number talk non-standardly suffices to rule out such nonstandard interpretation. Certainly, most people wouldn't say that it's part of what they mean by 'the natural numbers' that the natural numbers must have a structure which mirrors the structure of time (in the sense described above).

More importantly, requiring all acceptable interpretations of a person's number talk to make their cosmological hypotheses (like the one above) come out true suggests the following counterintuitive consequence. If time in our world had had a 'nonstandard structure'[13] then (even if we had no evidence of this fact) our number talk would have determinately referred to a nonstandard model. In contrast, my approach

[12]To bring out the weightiness of this assumption, note that physical theories on which spacetime is finite – in the sense relevant to Field's principle – have been seriously considered (c.f. [16] and[8] chapter 8). Perhaps one could even make sense of time having epochs structured like the nonstandard numbers.

[13]By talking about time having a nonstandard structure, I mean to consider a scenario in which the evenly spaced moments (e.g., the ticks of a perpetual clock) after a certain point form a nonstandard model of first order Peano Arithmetic.

will identify a more conceptually central aspect of our understanding of the natural numbers which would be violated if our natural number talk didn't refer to the standard model. And (as will become apparent) it does not suggest that, if the contingent physics around us had been undetectably different, our number talk we would have determinately referred to some nonstandard model[14].

4. My Proposal

With all this background in place, I will now argue for the core thesis of this paper: if we are are somehow able to give standard meanings to certain physical vocabulary and latch on to a notion of *physical possibility* (which has certain intuitively true features), this suffices to rule out nonstandard models of number theory. Thus, (if the relevant physical assumptions hold) merely grasping physical vocabulary and a notion of physical possibility provides a kind of back door to grasping the standard model of the natural numbers.

Remember that, as I outlined above, our goal is to identify a particular sentence which the speaker regards as in some sense definitional for the natural numbers, and then argue that (under some natural physical assumptions) the perverse interpret can only make this sentence come out true (while obeying the interpratational rules above) by interpreting the speaker's number talk standardly.

More specifically, the structure of this argument will be as follows:

[14]As we will see, my story risks failing to pin down determinate reference to the numbers, should its physical assumptions turn out to be false. Unlike Field's story, it doesn't seem to imply that we would have the wrong reference in this situation.

- Identify a claim COIN INDUCT which not only holds (physically) necessarily of the natural numbers but is sufficiently conceptually central to our notion of the natural numbers that we are willing to reject putative structures which fail to satisfy $\Box_P$COIN INDUCT (where $\Box_p$ is a physical necessity operator) as not what we mean[15] by the natural numbers. Specifically, we construct COIN INDUCT as the conjunction of the following (first-order English) claims

  HEADS INDUCT: An instance of the first-order induction schema as applied to the property $Q(n)$ asserting that there is an n-th coinflip and it landed heads.

  COUNTING RULES: A collection of quasi-analytic principles like 'For all objects x and numbers n, if x is the nth coinflip, then x is a coinflip.'
- Conclude that acceptable interpretations of the speaker's number talk must make $\Box_p$COIN INDUCT express a truth.
- Identify (in the meta-language) an intuitively true principle (IRS) about the nature of physical possibility which ensures that no non-standard interpretation of the natural numbers (i.e. a structure satisfying Peano Arithmetic) can make $\Box_p$COIN INDUCT true.

[15]At least in the weak sense that if some other structure does satisfy $\Box_P$COIN INDUCT without being otherwise unacceptable then the reference of the natural numbers must be one of those structures. We need not be so committed to this claim that we would be willing to totally abandon the natural numbers as an essentially incoherent concept should no structure satisfy $\Box_P$COIN INDUCT.

- Conclude that merely securing determinate reference to physical vocabulary and physical possibility ensures determinate reference to the natural numbers.

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Very crudely, one can think about my proposal as using the fact that it's physically possible for any subset of an infinite independent sequence of coinflips to land heads to take the place of second order quantification in ruling out nonstandard models of number theory. But now let's get into the details of the argument above.

4.1. **Assumptions.** Let's start by laying out some key assumptions needed for this argument (I will discuss some ways of weakening these assumptions in the next section).

First, we need a bunch of fairly bland assumptions which cash out the idea that we are taking reference to physical necessity for granted and impose some basic constraints on our interpreter. .

I will say that an interpreter is **well behaved** iff they satisfy the following criteria:

- They select a single model as the referent of our concept 'natural number' in all physically possible worlds.
- They don't tamper with extension of physical vocabulary like 'coinflip', 'landed heads' or 'temporally before' at any of these possible worlds.
- They interpret all first order logical vocabulary and the physical necessity operator standardly. So, for example, they will

take the speaker's existential quantifier and the physical necessity operator $\Box_p$ to contribute to truth conditions in the usual fashion.

- They interpret our natural number talk plausibly/charitably in the following sense: they interpret all sentences we endorse as conceptually central to our grasp and use of the natural numbers[16] (on par with the Peano Axioms) as expressing truths.

4.2. **Conceptually Central Claims.** To highlight the conceptually central sentences which my argument will appeal to, consider the way we use number talk when counting and referring to temporally (or otherwise) ordered sequences of events. For instance, we can talk about 'the 4th U.S. President', 'the 37th successful rickrolling', or 'the 3rd coinflip (in the history of the universe)' (Note that by 'coinflip' I will mean an objectively random process of the kind discussed in the next subsection.)

Given a conception of a type of physical events 'coinflips' we (and hence The Speaker) have or can easily supplement our language with[17] a relation 'countflip(n,x)' between coinflip events and events, which (intuitively) holds when 'x is the n-th coinflip' (counting from 0). We take this relation to relate 0 to the temporally first coinflip event in the

[16]Note that being conceptually central in the sense at issue here does not require being indubitable and could be a matter of degree [17].

[17]Since a normal English speaker can think 'there was no 17th ϕ' for arbitrary predicates ϕ, we probably ultimately want to analyze the logical structure of claims about the nth president etc. in a more generalizable way (via something like "x is the nth ϕ' iff there is a function from numbers to physical objects satisfying ϕ which respects the relation $>_\phi$ and maps x to n'). However I elide this complexity because it will make no difference to simple the model theoretic argument below.

history of the universe (if there is one)[18] 1 to the immediately temporally next coinflip event (if there is one) etc. Once using this notion we (and hence The Speaker) will take the following claims to be (something like) analytic truths which hold with physical and metaphysically necessity.

- An object x is the 0th coinflip, iff x is a coinflip and all other coinflips happen after x[19].
- If x is the nth coinflip, then y is the $n+1$th coinflip iff y occurs after x and no other coinflip occurs between x and y [20].
- Only coinflips can be the nth coinflip[21].
- No two distinct numbers correspond to the same coin flip [22].

Call the conjunction of the all the principles COUNTING RULES. Then it would seem that $\Box_P$ COUNTING RULES is (or follows from) principles treated as 'conceptually core' truths, which (just as much

[18]In some cases which coinflip counts as the temporally first one may be relative to a choice of a reference frame but this complication poses no difficulty. We may simply work relative to some foliation or, alternatively, restrict our attention to those possible worlds in which there is a unique first coinflip.
[19]That is, $(\forall x)[countflip(0,x) \leftrightarrow coinflip(x) \wedge (\forall y)(countflip(y) \rightarrow before(x,y) \vee x = y))]$. Here I depart from common practice in counting up from 0 rather than 1, for the sake of simplicity. Also by writing this formula with '0', I abbreviate the corresponding claim about the unique object satisfying some definite description of 0 in terms of $\mathbb{N}, S$, e.g., 'the unique object that has a successor but isn't a successor.'
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[20]That is,

$$(\forall n)(\forall x)(\forall y)(\mathbb{N}(n) \wedge countflip(n,x) \rightarrow [(countflip(S(n),y) \leftrightarrow coinflip(y) \wedge before(x,y) \wedge (\forall z)\neg (coinflip(z) \wedge before(x,z) \wedge before(z,y))]$$

[21]That is $(\forall x)(\exists n)(countflip(n,x) \rightarrow coinflip(x))$.
[22]That is $(\forall n)(\forall m)[coinflip(n,x) \wedge coinflip(m,x) \rightarrow m = n]$

as the Peano Axioms) all acceptable interpretations of our 'number', 'coinflip' and 'countflip' talk must get right[23].

Second, we (and hence the Speaker) take induction to hold for all predicates in our language, including $Q(n)$ "there is an n-th coinflip, and the n-th coinflip comes up heads," $Q(n) \stackrel{\text{def}}{=} (\exists x)(countflip(n,x) \wedge heads(x))$. And we can state an instance of the induction schema for $Q(n)$ as follows.

(HEADS INDUCT:)

$$Q(0) \wedge (\forall n)\,[Q(n) \rightarrow Q(S(n))] \rightarrow (\forall n)[\mathbb{N}(n) \rightarrow Q(n)]$$

Indeed we think its (logically, physically and metaphysically) impossible for any property to apply to 0 and the successor of every number it applies to without applying to all numbers. So we and the speaker will take HEADS INDUCT holds with physical necessity.

Let me thus define COIN INDUCT to be the conjunction of the induction schema for $Q(n)$ (the n-th coinflip landed heads) and the rules for counting coinflips. That is,

(COIN INDUCT:) HEADS INDUCT $\wedge$ COUNTING RULES

[23]Of course, the fact that COUNTING RULES is a conceptually core truth for us isn't to say that in some other context or language we couldn't instead start counting coinflips from 1 rather than 0. Also, it should be noted that these are only conceptually core principles for counting all coinflips. One could obviously count all coinflips after tuesday as well and we'd have to modify COUNTING RULES if we wished to describe counting those events.

By the remarks above I take it that $\Box_P$(COIN INDUCT) is (or follows from) principles which we (and hence the Speaker) treat as conceptually core. If the Putnamian skeptic allows that any acceptably charitable interpretation must make the axioms of Peano Arithmetic true it seems equally clear that any such interpretation must make $\Box_P$(COIN INDUCT) come out true. For this principle simply says that (an instance of) natural number induction and certain seemingly analytic truths about counting coinflips hold with physical necessity.

This completes my identification of the object language sentence ($\Box_P$ COIN INDUCT) which I will argue suffices to pin down our reference to the standard mode.

4.3. **Infinite Random Sequences.** I will now argue in the metalanguage that our perverse interpreter can't make $\Box_P$ COIN INDUCT true while obeying the rules above, without interpreting 'natural number' in a standard fashion. To this end I must make an assumption about the physical possibility of an infinite series of random events below.

> Infinite Random Sequence Assumption (IRS): It is physically possible to have a series of independent *objectively* random events with two possible outcomes and the following combinaton of features. The events are linearly ordered in time[24], there's a first event but no final event

[24]That is, for any distinct events x, y in the series, either x occurs before y or y occurs before x. Moreover, from the point of view of relativistic physics, the measurements are separated by time-like intervals (x is in the future light cone of y or vice versa) so all observers agree on their order. Given these constraints it is safe to simply work relative to some fixed inertial reference frame and ignore

and every event in the series has a temporal successor, i.e., for any event x there is some other event y occurring after x such that no event z occurs between x and y.

Informally, one can think of the events whose possibility IRS asserts as being like the ticks of an indestructible watch which never needs repair or winding. There is a first tick, each tick is followed by a unique next tick, and there is no tick after which the watch breaks down.

To motivate this principle, note that it is only asserting that it is physically possible to repeatedly perform (independent) textbook spin measurements on an electron[25][1] (or some other equivalent process) and that the laws of physics don't rule out time continuing infinitely into the future (though possibly having non-standard 'length'). I will abstract away from all details about these objectively random events in what follows, simply refer to each event as 'a coinflip' and the two outcomes as 'heads' and 'tails'[26]. [27]

relativistic complications for the remainder of the paper. Thanks to REDACTED for help developing this point and other details about physics.

[25]That is, perform a spin measurement along the x-axis on an electron whose spin has just been measured (and thus collapsed) along the y-axis.

[26]Even if you are not convinced that infinite random sequences of the kind discussed above are physically possible, they are surely plausible enough that it's uncomfortable for any philosophical view to be committed to ruling them out a priori.

[27]Also note that ultimately I will only use IRS to establish that it's physically possible for the events satisfying some physical predicate (having a determinate extension) in our current language to form an ω sequence (with the relation of 'temporally before' playing the role of $<$). So readers who already accept this claim could replace IRS with a direct assertion of the physically possibility of such a physically definable ω sequence. However, I feel IRS provides compelling reason to accept this conclusion.

Finally, note that mathematical vocabulary would naturally be used in stating the probability claims in IRS. As discussed in §2.2, I don't take this to be a problem for a realist aiming to defend herself from Putnamian skepticism. For I will only be appealing to the assumption that IRS (and using the mathematical vocabulary needed to state it) in metalanguage, to argue that that certain perverse interpretations of the Speaker are impossible.

That is, I don't assume that the Speaker can entertain the proposition IRS above (or even accepts the corresponding sentence). The Speaker just believes 'it's physically necessary that COIN INDUCT', and thereby forces the interpreter to make COIN INDUCT come out true at all physically possible worlds. We then use IRS and model theory to create a metalinguistic argument that certain worlds are physically possible and no interpretation (adhering to the constraints above) can interpret physical vocabulary standardly and number theoretic vocabulary nonstandardly at these worlds.

4.4. **Ruling out Nonstandard Models.** With these definitions in place, we can finally turn to foiling a (well behaved) perverse interpreter.

I will argue that, together with IRS, the above conceptual truths regarding counting ensure[28] that there is a physically possible world at

[28]Note that in many physically possible situations there will be a 'number' n such that these analyticities plus the facts about how $countflip()$, $coinflip()$ and $before()$ apply insure that there is no nth coinflip for certain values of n. For example, if no coinflips take place after the nth coinflip there will be no $n+1$th coinflip. Even in worlds whose possibility is asserted by IRS, it might be that there are only standard temporal durations, e.g., 'n-seconds after' only makes sense for standard integers n, in which case those worlds wouldn't have any n-th coinflip where n is non-standard.

which our current vocabulary picks out a counter-inductive collection of numbers[29](thereby witnessing that the restrictions on our interpreter should have prevented that choice of non-standard model).

First note that, by IRS, there is a physically possible world w where infinitely many independent random coinflips (linearly ordered by the relation '.. is temporally before..') take place. This sequence of coinflips need not itself form an ω sequence, but it has an initial segment which does[30]. By the independence and randomness of the coinflips, it follows that are physically possible scenarios corresponding to all combinatorially possible outcomes for each coinflip.

So there's some physically possible world where exactly an initial ω sequence of these coinflips come up heads. Now any nonstandard model of PA contains a proper initial segment which includes 0 and is closed under successor. I will argue that the above constraints on the perverse interpreter ensure that she takes $Q(n) \stackrel{\text{def}}{=} (\exists x)(countflip(n,x) \wedge heads(x))$ to hold for just those n in the standard initial segment of the nonstandard referent of the natural numbers – so that induction fails at this physically possible world for the property $Q(n)$.

Importantly, this argument will not question-beggingly presume that the speaker being interpreted can somehow pick out and refer to worlds at which only an initial ω sequence of coinflips come up heads. It only requires that the person being interpreted can secure definite realist

[29]That is, there is a physically possible world at which some property named in current English must be interpreted as as applying to 0 and the successor of every number it applies to, but not applying to all the numbers.

[30]Note that as the coinflips form an infinite discrete linear ordering with least element (by IRS), it follows that they include an initial ω sequence.

reference for the physical vocabulary mentioned above $coinflip$, $heads$, $before$ and $\Box_P$, and that they treat the following claim as conceptually necessary.

> It's physically necessary that the natural numbers satisfy: the first order axioms of number theory, induction on the property '*there is an nth coinflip and it comes up heads*' and COUNTING RULES.

I claim the perverse interpreter can't simultaneously satisfy the induction axiom for $Q(n) \stackrel{\text{def}}{=} (\exists x)(countflip(n, x) \wedge heads(x))$ and COUNTING RULES above (while taking $coinflip()$, $heads()$ and $before()$ to have their intended extension) at this troublesome world w. To see why, imagine her predicament when choosing an extension for '$countflip()$' at this world, where exactly the initial ω sequence of coinflips comes up heads. The principles governing '$countflip()$' tell us that 0 has to be assigned to the temporally first coinflip in w, 1 to the next, and so on for all the objects in the standard initial segment of the nonstandard model. This 'uses up' all the coinflips landing heads. Since, by our principles, no coinflip can be counted again (i.e. associated with another putative natural number) it follows that all and only the 'true' (i.e. standard) natural numbers are paired with coinflips landing heads. So the perverse interpreter must make '$Q(0)$' and 'whenever Q applies to some number n it also applies to $Q(n + 1)$' true. But as the perverse interpreter takes 'the numbers' to refer to a larger structure than this standard initial segment, their interpretation also makes '$\forall n Q(n)$'

false. Thus, their interpretation renders COIN INDUCT false at this world and thus $\Box_p$ COIN INDUCT false as well.

So (to summarize) if IRS is true than (provided coinflip(), heads(), before() have their usual interpretation) either 'natural number' is interpreted standardly or the induction schema doesn't hold with physical necessity.

5. Generalizing the point by dropping/changing some assumptions

So much for my core argument that any answer to model theoretic worries about determinate physical reference would thereby also block analogous worries about number theoretic determinacy. I have argued that (if a certain plausible physical assumption holds) one cannot give a nonstandard interpretation for our number talk while interpreting physical vocabulary standardly and making certain apparently obvious and conceptually central principles come out true.

Now let me point out some quick generalizations and corollaries to the main argument above.

First, the argument above works just the same if we replace appeal to physical possibility $\Box_p$ with appeal to metaphysical possibility $\Box_m$. And even if one doubts the physical possibility of an infinite random sequence of independent coinflips (as per IRS) it is hard to doubt the *metaphysical* possibility of such a sequence. So if we are talking to a philosopher who allows that we can grasp a fairly definite notion of metaphysical possibility, we can drop the weighty assumption that infinite series of independent objectively random events are physically possible, in favor of the assumption that we can grasp a suitable notion of metaphysical possibility (and the very widespread assumption such independent objectively random events are metaphysically possible)[31].

Second, we don't need to assume possession of a completely determinate notion of physical/metaphysical possibility. It's fine if our concept of physical possibility is under-specified in a number of different ways – provided that all acceptable precisifications agree in classifying the infinite sequence of objectively random events invoked above (and hence the relevant possible world w_ω) as physically possible. And this is something which many existing reasons for thinking that our notions of physical/metaphysical possibility aren't fully definite don't call into question.

Third, we might even be able to drop the assumption of definite reference to a notion of physical/metaphysical possibility, if we think think principles of charity[3] favor making speakers out to be rational/justified as well as attributing them more true beliefs. For, (arguably) if we don't interpret people to be talking about some the standard model of the natural numbers, then their (approximately) *a priori* confidence in instances of the induction schema will look like unjustified dogmatism, in the following sense. The existence of physical ω sequences as in world w_ω, seems like something we shouldn't rule out

[31]We could also make a similar argument without appealing to the notion of random events at all. We could just appeal directly the intuition that (from a realist point of view) it would be metaphysically possible for there to be an ω sequence of coinflips which came up heads. This seems strongly motivated by the intuition that infinite collections of some kind are physically possible plus general Humean Recombination intuitions.

a priori. And (by the argument above) any nonstandard interpretation of our number talk would make induction fail in this epistemically possible scenario. Thus our great a priori confidence in $\Box_p$COIN INDUCT (and our disposition to hold to this belief, even if we come to think there might be infinitely many coinflips), looks rational on a standard interpretation of the numbers, but irrational on any nonstandard interpretation of our number talk.

6. Conclusion

In this paper I have argued against the combination of rejecting determinate reference to the numbers on Putnamian model theoretic grounds while accepting determinate physical reference. Specifically, I have argued that (given certain mild assumptions), merely securing determinate reference to physical possibility and certain other physical concepts suffices to rule out all non-standard models of our number theoretic talk as well.

Let me close with three notes of humility. First (as noted above), I don't claim to have answered Putnam's challenge, or explained how we can determinately grasp mathematical concepts. I have only argued for a (dialectically important) conditional claim. If one could somehow answer Putnamian's challenge regarding physical concepts this would provide a principled reason for rejecting nonstandard interpretations of our number talk as well. I've provided no argument that we should respond to this fact by rejecting Putnam's against physical and number theoretic determinacy alike rather than, e.g., taking them to succeed in both cases.

Second, remember that the Putnamian challenge sought to identify an internal problem for the realist which alleges that, even on their own view, they can't explain how we possess definite reference to structures like the natural numbers. In this paper I've shown that, if we assume determinate reference to physical objects and possibility, this challenge can plausibly be answered. However, this doesn't show that fans of determinate physical reference can't deny we have a determinate concept of the natural numbers *on some other grounds than Putnam's model theoretic challenge*.

Third, although I have argued that (if we can somehow definitely refer to it) leveraging physical possibility offers a route to grasping a definite concept of the natural numbers, I don't mean to suggest that this is the only or primary way that we can grasp such a concept. It would be strange if our possession of a definite conception of the natural numbers depended on our beliefs about physical (or metaphysical) possibility. Thus, I suspect that another - rather different- style of answer to Putnam's challenge must also be possible.

References

[1] David Z. Albert. *Quantum mechanics and experience*. Harvard University Press, 2009.

[2] Tim Button and Sean Walsh. Structure and Categoricity: Determinacy of Reference and Truth Value in the Philosophy of Mathematics. *Philosophia Mathematica*, 24(3):283–307, 2016.

[3] Donald Davidson. Three Varieties of Knowledge. In A. Phillips Griffiths, editor, *Royal Institute of Philosophy Supplement*, pages 153–166. New York: Cambridge University Press, 1991.

[4] M. Devitt. PUTNAM, H., "Meaning and the Moral Sciences". *Australasian Journal of Philosophy*, 58(n/a):390, 1980.

[5] Hartry Field. Which undecidable mathematical sentences have determinate truth values? *Truth in Mathematics*, pages 291–310, 1998.

[6] Hartry Field. *Truth and the Absence of Fact*. Oxford University Press, 2001.

[7] K. Gödel. Über formal unentscheidbare sätze der principia mathematica und verwandter systeme i. *Monatshefte für Mathematik*, 38(1):173–198, 1931.

[8] S. Hawking. *A Brief History of Time*. Random House Publishing Group, 2011.

[9] David Lewis. New Work for a Theory of Universals. *Australasian Journal of Philosophy*, 61(4):343–377, 1983.

[10] David Lewis. Putnam's Paradox. *Australasian Journal of Philosophy*, 62(3):221–236, 1984.

[11] Vann McGee. How We Learn Mathematical Language,. *The Philosophical Review*, 106, 1997.

[12] Charles Parsons. The Uniqueness of the Natural Numbers. *The Jerusalem Philosophical Quarterly*, 39:13–44, 1990.

[13] Hilary Putnam. Realism and Reason. *Proceedings and Addresses of the American Philosophical Association*, 50(6):483–498, 1977.

[14] Hilary Putnam. Models and Reality. *Journal of Symbolic Logic*, 45(3):464–482, 1980.

[15] Jared Warren and Daniel Waxman. A Metasemantic Challenge for Mathematical Determinacy. *Synthese*, pages 1–19, forthcoming.

[16] Wikipedia contributors. Big crunch — Wikipedia, the free encyclopedia, 2019. [Online; accessed 3-March-2019].

[17] Timothy Williamson. *The Philosophy of Philosophy (The Blackwell / Brown Lectures in Philosophy)*. Wiley-Blackwell, Oxford, 2008.