Explanatory Indispensability and the Set Theoretic Multiverse

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Abstract

Width multiverse approaches to set theory (like Joel David Hamkins' influential proposal in [5]) reject the idea that there's an intended width hierarchy of sets which contains 'all possible subsets' of the sets that it contains. In this paper, I raise an explanatory indispensability worry for the multiverse theorist and distinguish three different possible styles of response to this worry. I will argue that each approach faces some serious prima facie problems. And I'll suggest that, by clarifying their response to this puzzle about applications, multiverse theorists can helpfully clarify their proposals concerning pure mathematics.

1 Introduction

On a conventional understanding of set theory, there's a unique intended hierarchy of sets that contains, at each layer, sets corresponding to 'all possible ways of choosing' sets from lower levels. This determines a unique intended right answer to all set-theoretic questions, like the continuum hypothesis (CH), whose truth value only depends on the width of the hierarchy of sets.

In contrast, what I will call width multiverse theories (influentially exemplified by Hamkins in [5]) agree that there are Platonic mathematical objects, the sets, but deny that there's a unique intended hierarchy of sets (even up to width). Instead, they take there to be a multiverse of different hierarchies of sets (set theoretic universes). And they maintain that there's no general intended right answer to certain set-theoretic questions whose truth value varies between universes. Rather mathematicians simply choose, in different contexts, to work in one kind of universe or another. In particular, the width multiverse theorists I'll be concerned with in this paper (henceforth, I will just call them 'multiverse theorists') accept the following claims.

- Sets literally exist
- For every set-theoretic universe V, there is a strictly wider universe V[G] corresponding to (what is called) a forcing extension of V. This V[G] contains all the sets in the original universe but adds extra subsets of sets in the original universe.
- There's no context-independent right answer to questions like CH (the continuum hypothesis) whose truth value varies between set theoretic universes.

In this paper, I will discuss a challenge for multiverse theorists generally, and especially for Hamkins – who advocates a particularly bold form of multiverse theory which also denies there's a unique intended natural number structure, unlike most multiverse theorists.

Crudely, the explanatory challenge I have in mind goes like this. When giving scientific explanations, we currently are open to hypotheses (traditionally stated via talk of sets) which invoke facts about 'all possible ways of choosing'. But accepting width multiverse theory, seems to require rejecting this notion (or at least denying traditional claims about how it connects to set theory). Thus, accepting width multiverse theory threatens to commit one to ruling out seemingly cogent candidate explanations for physical facts, a priori.

In this paper, I will develop the above worry and consider various ways Hamkins and other width-multiverse theorists could try to answer it.

In §2 I'll clarify what I mean by a lazy explanatory indispensability argument and develop a particular lazy explanatory indispensability argument against multiverse theorists. In the remaining sections, I'll discuss three styles of response to this worry. In §3 I'll consider responses which claim some particular set-theoretic universe V_p within the multiverse capures constraints on how physically definable properties can apply that apply as a matter of physical law. In §4 I'll consider responses which take facts about all possible ways of choosing to be reflected by facts about what sets exist within the multiverse as a whole. And in §5 I'll consider approaches that replace appeals to all possible ways of choosing with appeals to first-order logical deduction (in various ways).

I will argue that each approach faces some important difficulties. However, my aim is not to refute width-multiverse theory. Rather, I hope to show that there's a legitimate concern here (arising from traditionally expected relationships between set theory and logical possibility), which allows for a range of importantly philosophically different possible responses. Multiverse theorists like Hamkins can helpfully elucidate their views about pure mathematics by clarifying their favored response to this challenge.

2 The Explanatory Indispensability Worry

So, let's begin with the very idea of a lazy explanatory indispensability argument. Classic explanatory indispensability arguments [12, 10, 1, 3] against mathematical nominalism maintain that we should accept mathematical objects, because the physical theories which best predict and explain certain empirical data can't be formulated without quantifying over them. But in this paper, I'll present a slightly different type of challenge which differs from the above classic explanatory indispensability arguments in two ways.

First, the argument I'll develop attacks mathematical truth value antirealism, not mathematical object antirealism. It doesn't argue that *mathematical objects* are needed to give certain kinds of intuitively good scientific explanations. After all, width multiverse theorists like Hamkins already accept the existence of plenty of sets! Rather it argues for accepting more *truth-value realism* than the multiverse theorist currently does.

Second, the argument I'll consider is (what I'll call) a lazy explanatory indispensability argument in the following sense. It poses an a priori rather than a posteriori challenge. Unlike classic explanatory indispensability arguments, it doesn't try to point out a part of our actual best scientific theory that advocates of the view being criticized (in this case multiverse theory) can't adequately express, or data they can't adequately explain. Instead, it suggests that multiverse theory implausibly rules out certain seemingly-cogent physical hypotheses a priori.

To see what I mean, consider a classic explanatory indispensability argument, like Baker's argument that nominalists can't reproduce the explanatory power of Platonist explanations for prime length life cycles of cicadas. Imagine learning that the evidence Baker cites for cicadas having prime length life cycles was a hoax. Would this fully quash Baker's challenge to mathematical nominalism? Not necessarily. Many would feel that inability to adequately capture this hypothesized scientific explanation still revealed a problem for mathematical nominalism. For they would be hesitant to accept any philosophy of mathematics that required us to stop considering such explanations as a live option.

In the next subsection, I will argue that Hamkins and other multiverse theorists face an analogous worry. Accepting multiverse theory threatens to imply implausible restrictions on the space of candidate physical explanatory hypotheses. Admittedly, this kind of lazy explanatory indispensability argument likely won't interest very strongly empiricist philosophers. However, that doesn't prevent it from being quite significant and troubling for those who don't have hangups about the a priori.

2.1 Core Worry

So now let's turn to developing the specific lazy explanatory indispensability argument against multiverse theories at issue in this paper. This worry arises from traditionally expected connections between set theory and a notion of logical possibility/'all possible ways of choosing' which can be used in physical explanations. It raises the question: does multiversist truthvalue antirealism about set theory force one to accept uncomfortable antirealism about all possible ways of choosing as well?

From a traditional point of view, we seem to have a modal notion of 'all possible ways of choosing', which acts like a bridge between mathematical and non-mathematical reality, in the following interesting way. Facts about all possible ways of choosing are expected to have close a priori connections to both counterfactual-supporting constraints on non-mathematical reality (on the one hand) and set theory (on the other). In particular

- Facts about 'all possible ways of choosing' are supposed to connect to set theory, by helping determine the unique intended structure of the set-theoretic universe (up to width). For each layer of the iterative hierarchy of sets is supposed to contain sets corresponding to all possible ways of choosing some sets that occur at lower levels of the hierarchy.

- Facts about all possible ways of choosing are also supposed to constrain non-mathematical reality, so that appeal to them can help predict and explain regularities involving physical objects. For example, if there's no possible way of choosing colors for countries on a certain physical map such that no two adjacent regions have the same (literal physical) color, we think three things follow. First, the map isn't actually three-colored. Second, the map couldn't 'easily' have been three-colored (i.e., it isn't three-colored at any close possible worlds)[1]. Third, what rules out the map actually being three-colored is a general logical/combinatorial constraint (i.e., one that applies analogously to all predicates and relations). So it follows that the map isn't (and couldn't easily have been) three-scented or three-textured either.

Because of this traditionally-expected bridge between set theory and lawlike constraints on non-mathematical reality, claims about set with ur-elements can be used to explain (or evoke modal principles that explain) physical regularities, in cases like the following[2].

Suppose we have a (finite or infinite) physical map[3] which has never been three-colored, despite many changes in the colors of individual map regions. That is, suppose there's never been a point at which each map region is either red, green or blue but no two adjacent map regions have the same color.

Prima facie, a true and illuminating explanation for why the map isn't three colored might be that it's not three color*able*, in a modal sense (reflected by the

[1]Plausibly all very close possible worlds preserve the way that map regions are related by adjacency, and relevant logico-combinatioral constraints apply with metaphysical necessity so the map isn't three colored at these close possible worlds either

[2]Here I highlight the intuitive legitimacy of such explanations involving infinitely many physical objects, because some might try to argue that multiverse theorists can more easily allow there's a favored notion of all possible ways of choosing from finitely pluralities (via the idea that forcing extensions don't change these facts). This presumption will be somewhat challenged below.

[3]Presumably, there aren't really any infinite physical maps. Perhaps one could make the cases more realistic by appealing to infinitely many galaxies (or points in physical space or point particles) with adjacency relations between them. However, I won't attempt to do that here. Instead, I'll merely appeal to the apparent concievability of certain scenarios involving mathematically explained regularities concerning physical maps, to give a lazy explanatory indispensability argument in the sense described above.

fact that there's no set coding a three coloring function in the hierarchy of sets with ur-elements).

> **Infinite Map Non-Three Coloring Explanation** (traditional universe-ist version): The map isn't 3-colored because there is no set coding a 3-coloring function and, since every possible way of choosing is witnessed by a set, if there were a way of 3-coloring the map there would be such a set.

But what happens to this picture if we adopt a multiverse understanding of set theory? As noted above, the multiverse theorist must deny that there's a 'full width' hierarchy of sets which contains all possible subsets of sets it contains[4]. So it seems they must either

- Reject the above notion of 'all possible ways of choosing' or
- Accept this notion but say that (for some reason) no single hierarchy of sets can contain all possible subsets of sets it contains.

Thus, the multiverse theorist faces a question about seemingly cogent physical explanatory hypotheses which appeal to this notion of all possible ways of choosing (via set theory). It's hard to deny that something like (some version) these claims express a legitimate explanatory hypothesis. But it is not clear how to make such physical explantions compatible with multiverse set theory.

Thus, we get a lazy explanatory indispensability argument.

Before discussing different styles of response to the above challenge in the remainder of this paper, let me note two things.

First, note that worry here doesn't just arise when different universes in the multiverse are supposed to *disagree* on the truth-value of mathematical claims

[4] For, they think each universe V exists alongside an expanded universe V[G] which adds 'missing subsets' of some sets that are already in V.

a physical explanatory hypothesis[5]. Rather it applies to all explanations which assume a connection between set theory and law-like constraints on physical objects. As we have seen, the multiverse theorist rejects the traditional bridge between set theory and non-mathematical reality (the assumption that facts about sets reflect general law-like constraints on 'all possible ways of choosing' which constrain how any relations can apply to physical objects). But once we demolish this bridge, it becomes prima facie unclear why the non-existence of certain sets (e.g. sets coding a three coloring) — in one universe, or the multiverse as a whole — should imply *anything* about how physical properties can apply to physical objects.

Second, the physical explanatory hypotheses we need to account for can have different logical forms and properties. For example, we might say the three coloring explanation invokes a kind of $\neg\exists$ claim (that there's no set with a certain property). But other seemingly cogent explanatory hypotheses have a more complex structure ($\forall\exists$ rather than $\neg\exists$), like the following.

> **Troop Distribution**: The reason why no one has succeeded in holding such-and-such map region is that, for every possible way of stationing defending troops in countries on the map satisfying ... constraints, there's a way of stationing attacking troops such that ...[6]

With this in mind, let's turn to some possible answers to the above challenge. I will discuss three strategies for responding to the above lazy indispensability challenge – by formulating a version of relevant seemingly cogent physical

[5]In fact, as we will see in §5, there are special reasons the multiverse theorist should say all universes agree that there's no three coloring function, in cases where a map intuitively isn't three colorable. But these reasons don't apply to other seemingly cogent physical explanatory hypotheses like the troop distribution example below.

[6]And perhaps we can cash this out more realistically by saying for each way of initially placing some FOO-particles in space, there's some way of placing BAR-particles such that, or some stable such and such will form...)

explanatory hypotheses that's compatible with multiverse theory.

3 A Physically Preferred V_p

The first strategy I want to consider takes inspiration from Hamkins' remark that there are surprisingly deep analogies between his favored approach to set theory and (a certain version of) common contemporary pluralism about geometry[5].

One might argue that mathematical and/or physical discoveries in the early 20th century showed we should separate physical geometry from mathematical geometry by taking the following position. Many different choices of geometrical axioms are equally legitimate topics for a priori investigation. However, there's a physically correct geometry – a choice of geometrical axioms which reflects the true structure of physical space and thereby certain law-like (physically necessary) constraints on the behavior of physical objects.

Appeal to facts about this physically favored geometry can take over the traditional role of appeals to the one true geometry in physical explanation. So, for example, claims about physical geometry/the structure of physical space can replace appeals to traditional geometry in explaining in why round manhole covers are useful, why certain patterns in the area and side-lengths of various tracts of land consistently obtain and the like. Thus there is no serious loss in physical explanatory power from switching from traditional to pluralist geometry, and many of our old geometrical explanations for physical facts can be retained with only small revisions.

Inspired by this, a multiverse theorist might take an analogous approach to physical explanations which appeal to traditional single universe set theory. In this case, they would say that there's a physically special universe of sets with ur-elements V_p, with the following property. Physical law prevents either initial conditions or, say, the results of physically random events from ever letting physical properties apply in a way that isn't witnessed by a set in V_p.[7]

The multiverse theorist could then reformulate physical theories and explanations (like the three coloring explanation above) by replacing traditional appeals to facts about this physically special V_p. So, for example, they might render the three-colorability explanation above as follows.

> There's a certain physically preferred set theoretic universe V_p within the multiverse, which reflects lawlike constraints on how *all physically definable properties* can apply to actually existing objects, in the following sense. For all existing objects xx and physically definable property ϕ, it would be physically impossible for ϕ to apply to some yy among the xx, without V_p already (actually) containing a set with exactly these objects yy its elements.

[7]Specifically, the multiverse theorist might say that it's a **physical law** that physical properties can't apply to physical objects in a way that would let some formula $\phi(x, o_1, ..o_n)$ (with only physical properties and logical relations in ϕ and only physical objects $o_1 \ldots o_n$ – or perhaps sets in V_p – as parameters) pick out a set of physical objects that isn't already in V_p, as follows.

- **Physical Comprehension**: if x is a set in V_p, then V_p also includes all 'sufficiently physically definable' subsets of x. So, for example, it satisfies comprehension for all English formulas ϕ with parameters ranging over physical objects and sets in V_p, but not other sets in other universes in the multiverse. Here are some examples of what this principle requires.
  - Since (by the ur-element principle U) V_p contains a set of all physical objects, this principle tells us V_p must contain a set of red physical objects (and green ones, positively charged ones etc.).
  - If V_p contains a pure set x (say, its version of the numbers) and a function f from the numbers to the marbles (e.g. f(n) might be the n-th marble you've seen in your life), then this principle tells us that V_p must also contain a set of the elements of x such that f(x) is a red marble.
- The above claims holds with physical necessity. That is, the laws of physics prevent any physically definable property from picking out a missing subset of V_p (i.e., applying to some physical objects in a way that is not *already* 'witnessed' by a set that actually exists in V_p).

Someone who takes this approach might also take this V_p to be physically special in the following way. Whenever traditional realists about set theory would say physical law allows possible outcomes corresponding to all possible ways of choosing from some physical objects (e.g., in scenarios involving infinitely many QM random independent coinflips), all sets with ur-elements in V_p must correspond to genuine physical possibilities (e.g. physically possibilities for which coinflips come up heads).

There is no set witnessing a way of three-coloring a map in this physically preferred V_p.

Therefore the map isn't three-colored – and, indeed, it would be physically impossible for it to be three-colored (while facts about how map tiles are related by adjacency are held fixed)

Note that the restriction of the above-hypothesized law about V_p to properties which are *physically definable* is not optional. The multiverse theorist might have wanted to mirror conventional set theory better by proposing a physical law that V_p contains sets of physical objects corresponding to the extension of every property definable with parameters[8]. However, they can't say this. For, the multiverse takes V_p (like every universe) to have a universe corresponding to a forcing extension $V_p[G]$ which adds missing subset G of the natural numbers in V_p. But if there is an infinite collection of physical objects xx[9], we can use G as a parameter to define plurality yy from among these physical objects xx, such that V_p doesn't contain a set corresponding to the yy.[10]

This approach has some appeal. Aside from the parallel with geometry, it lets the multiverse theorist preserve the intuitive counterfactual-supporting

[8]From a naive/traditional point of view, facts about set theory constrains the physical world because V contains 'all possible subsets', in a way that ensures for each set z in V, V includes all subsets of z which can be defined in the following ways:

- using **any** objects as parameters (not just sets in V) and any relations in our language (not just ∈)
- using any variant language we might speak in contexts where we add new predicates or names **or drop quantifier restrictions**[9]

So we accept the following schema as holding with metaphysical necessity:

Full Comprehension Schema (I slightly abuse notation, for legibility, in writing $z \in V$) for V $\Box(\forall z \in V)(\forall w_1)\dots(\forall w_n)(\exists y \in V)(\forall x)[x \in y \Leftrightarrow ((x \in z) \wedge \phi)]$

The width multiverse theorist might want to mirror this claim, but say that it's physically necessary that V_p has the properties ascribed to V in the Full Comprehension Schema. However (for reasons discussed in a footnote below), it turns out one cannot.

[9]By this I mean, 'if V_p contains a function f mapping its copy of the natural numbers to these physical objects in a 1-1 way'.

[10]By using the relevant function f and this generic G as parameters, we can define a property (being in the image of G, under f) whose extension cannot be in V_p. For, by the assumption that V_p satisfies basic axioms of set theory with ur-elements like ZF, if it contained a 1-1 function of f and the image of G under f, it would have to contain G. Positing a physical law which only constrains how *sufficiently physically definable* properties apply lets us get around this problem.

force of traditional explanations. And applying it is straightforward; we just replace appeals to rejected logical/combinatorial facts about 'all possible ways of choosing' (which would have to apply to mathematically and physically defined properties alike) with appeals to more limited physical laws.

However, if we take this approach, the question, 'how does physics control the outcomes of seemingly random independent events (e.g., coin tosses), to avoid realizing (i.e., letting us use physical vocabulary to define) a missing subset?' can be troublesome. Maybe it's just a brute physical law that the outcomes of coinflips and painting countries etc. always avoid letting one define the missing subset. But this can be awkward. For is it plausible that a physical law steps in to prevent all physical properties from applying to relevant pluralities of physical objects yy? Arguably the concept of physically definable properties is too unnatural to figure in a plausible fundamental physical law.[11]

4 Appeal to the Whole Multiverse

Now let's turn to a different response to the lazy explanatory indispensability worry above.

The multiverse theorist might allow that there are genuine (and fully determinate) facts about all possible ways of choosing from some physical objects, which constrain any properties can apply, but deny that any single set-theoretic

[11]One might try to avoid the problems above by simply stipulating, that in applied mathematical contexts, we always mean to talk about a set-theoretic universe that happens to satisfy the Physical Comprehension principle above (i.e., a V_p that contains all subsets of sets it contains that are physically definable, given how physical properties *happen to actually apply*). But such an approach wouldn't offer any genuine explanations, only dormative virtue non-explanation (the maps will never be three-colored because it will never be three-colored).

If we took this approach, the that our contextually relevant V_p does not contain any set witnessing a three coloring would indeed *imply* that the map wasn't three colored. But it would not explain the latter fact. For this deduction as an explanation for why the set never actually got three colored would be like saying, 'The reason why Jake doesn't have a driver's license is that the list of all people who hold a driver's license doesn't include Jake'. Neither of these would-be explanations preserves the intuitive counterfactual supporting force of our original explanation. They don't rule out the possibility that their explanandum is a complete fluke: that that map could very easily (at very close possible worlds) have been three colored or Jake could very easily have learned to drive.

universe can witness all possible ways of choosing in the way the traditionally intended hierarchy of sets is supposed to[12]. Rather, (they will say) *the multiverse as a whole* contains sets witnessing facts about 'all possible ways of choosing' some physical objects.

Accordingly, we can replace traditional physical explanations which appeal to sets in the unique intended set-theoretic universe V with corresponding claims that quantify over all sets in all universes in the multiverse. For example, we will re-write the sample non-three coloring explanation as follows.

> No universe anywhere in the multiverse contains a set three coloring the map — and the multiverse contains sets witnessing all possible ways of choosing. Thus, the map isn't three-colored.

I personally think this style of response runs counter to the spirit of many width multiverse theories, and especially Hamkins' proposal. However, in the remainder of this section, I will present a more concrete objection. In the main text, I'll argue that the assumptions needed for this proposal would let us talk about a unique intended natural number structure (contra Hamkins). And in appendix B I'll show how the same techniques can be used to recover an intuitively intended truth value for CH, contra width multiverse theorists' antirealism about CH.

Abstractly, the problem is this. If we assume the multiverse contains sets witnessing all possible ways of choosing from some physical objects (and that we can quantify over all such sets when giving physical explanations), we get all the expressive power of traditional second-order quantification over physical objects. This creates a problem for Hamkins, because it lets us write a sentence (call it 'Physical ω Sequence'), which implies that some physical objects considered under some physical relation (e.g., the stars that have a certain

[12]That is, there is and can be no single universe which contains at each layer sets corresponding to all possible ways of choosing some sets at lower layers.

property, considered under the relation 'is farther from the sun than') form an intuitively intended model of the natural numbers. Accordingly, we can (contra Hamkins) talk about intrinsically favored/intended natural number structure, as the structure the relevant physical objects would have in the metaphysically possible scenario of Physical ω Sequence being true – regardless of the fact that differently structured models of PA play the role of 'the natural numbers' in different set-theoretic universes.

To flesh this argument out, note that a multiverse theorist employing the strategy considered in this section should accept the following principles.

- Generalization to Ur-elements: There's (not just a multiverse of pure sets but) a multiverse of hierarchies of sets with ur-elements.

- (Quantification over) Whole Multiverse: We can unproblematically quantify over all sets in every universe in the multiverse (at the same time).

- Plentitude: For all possible ways of choosing tuples of physical objects, there is a universe in the multiverse and a set in that universe containing exactly those n-tuples.

- Metaphysical Necessity of Plentitude: The above plentitude claim holds, not just at the actual world, but at all possible worlds. That is (speaking in terms of possible worlds), at each metaphysically possible world w, the physical objects exist alongside a multiverse of universes of sets with-ur elements which is plenitudinous in the sense above.[13]

  So, for example, if Socrates had had three extra siblings, there would have been (somewhere in the multiverse) a singleton set containing each of these siblings, and sets witnessing all ways of choosing some of them

[13] That is, for all pluralities xx of physical objects in w, the multiverse in w contains some universe with a set whose elements are exactly the xx. And the same goes all possible ways of choosing n-tupples of physical objects (i.e., all possible ways an n-place relation could apply to physical objects in w) being witnessed by some set of sets coding n-tupples in some universe in the multiverse at w.

I will show that, under these assumptions, we can write down a statement which (is intuitively metaphysically possible and) implies that the stars form an intended model of the natural numbers. This claim may not be true of the actual world. However, it lets us talk about a unique *naively intended* natural number structure, as the structure the stars would have to have in the metaphysically possible scenario where this description holds true. And this in turn lets us produce if-thenist paraphrases which capture the intuitively intended truth conditions for any number-theoretic sentence ϕ.

Now let's get into details. I take it to be intuitively metaphysically possible (and perhaps actually true that) that some physical objects satisfy PA_{-} (i.e., the finitely many Peano Axioms for arithmetic, sans the induction schema) when considered under some physical relation. For example, there could be stars located such that PA_{-} comes out true when we replace 'number' with 'star' and 'successor' with S_p 'is next furthest from the sun after'[14]

And we can write down a property ψ which picks out an initial segment of any structure of stars which forms an intuitively intended model of the natural numbers, as follows.

> $\psi(x)$ iff x is a star and for every set S in some universe in the multiverse, S is successor closed (in the sense that for all stars y, if $y \in S$ then the star that's next further from the sun than y is also in S), then x is in S.

Given our current assumptions (about the multiverse as a whole containing sets corresponding to all possible ways of choosing some physical objects), if the stars satisfy PA_{-}, then the stars satisfying ψ will form an intuitively intended model of the natural numbers.

[14]Note that this just requires things like there being a closest star to the sun, and every star having a successor star (in the sense above).

Thus we can describe a metaphysically possible situation where some physical objects, considered under some relation (e.g., the stars with property ψ considered under the relation 'is further from the sun than') would have to form an traditionally intended model of axioms for number theory (a genuine ω sequence).

And this provides us with something Hamkins wants to reject: a way to specify a unique intended natural number structure (as that structure which the stars satisfying ϕ would have in the metaphysically possible scenario described above).

What about our more moderate-width multiverse theorist, who happily accepts the existence of a unique intended natural number structure, but denies there's an axiom choice independent right answer to CH (and other questions whose truth value can be changed by forcing)?

In appendix B, I'll discuss how we can use the same method (using quantification over all universes in the multiverse to simulate second order quantification) to create a problem for multiverse theorists generally. I'll show that we can write down a sentence which (given the three assumptions above) is true iff CH is false (when relevant notions of powerset and cardinality are cashed out in terms of all possible ways of choosing in an intuitive way).

5 Appeal to Provability in FOL

The final style of response to our lazy explanatory indispensability worries I want to consider, tries to replace claims about what sets exist with claims about provability in physical explanatory hypotheses like the three coloring explanation in §2.

This approach can be motivated by considering cases like the simple non-three-coloring explanation from §2 and noting the following. Whenever the traditional non-three colorability claim is true of some map, there's an alternative explanation which deduces the fact that the map isn't three-colored from finitely many physical facts about which countries on the map are adjacent to each other and (robust, counterfactual supporting) first order logical laws. [15]. Thus (from a traditional point of view), if the non-three colorability explanation is true, there's a version of the explanation that eliminates all appeal to sets and all possible ways of choosing. Specifically, there's a first order logical proof that the map isn't three colored from concrete first order logical facts about which map regions are adjacent to each other.

I think the main problem with this strategy as an answer to the lazy indispensability challenge I've raised is that it doesn't generalize (in any obvious way) to handle other seemingly cogent physical explanatory hypotheses that invoke a notion of all possible ways of choosing/set theory. For example, remember the hypothesis that some map regions had never been held for long because, 'for every possible way of choosing a defending troop disposition which..., there's a possible way of choosing an attacking troop distribution which...'. Unlike in the three coloring case, there's no appearance that this explanation is true if and only if some concrete first-order logical statement (i.e., one that avoids set theory, second-order logic, claims about all possible ways of choosing or the like) about adjacency and map regions entails some other

[15] By the compactness and completeness theorems, in each particular case where a map isn't three-colorable, there will be some true non-mathematical, purely first-order logical sentence about the countries which implies that if every country is either red, green or blue, two adjacent regions have the same color. For consider an infinite first order language with names for every map region and the relations 'is a physical map region', 'is adjacent to'. The set of all truths in this language uniquely pins down the structure of the physical map regions under adjacency. So if the map isn't three colorable then (by completeness), contradiction can be derived from conjoining the set of all truths in this language with the claim that the map is three-colored (i.e., every region is either red, green, or blue and it's not the case that there are two adjacent regions that are both red, both green or both blue). And (by the fact that proofs can only use finitely many premises) there's some conjunction of finitely many sentences that truly describe the map in our infinite language which logically entails that the map is not three-colored. Hence (by considering a Ramsey sentence which uses quantification to eliminate all names) there's a true sentence (in a language without these extra names for map regions) that concretely describes the adjacency facts about the map and logically implies that the map is not three colored.

concrete first order logical statement about the map.

However, I will now discuss worries about whether Hamkins (or other multiverse theorists) can use the above strategy to simulate/replace *even* the basic three-coloring explanation above. Although I think failure to generalize is a big unavoidable problem for the general strategy of replacing set theory claims with provability claims, I'll discuss these specifics because I think they raise interesting philosophical choice points for Hamkins and the multiverse theorist.

In particular, we face a dilemma when trying to cash out our three coloring explanations in terms of provability.

On one hand, the multiverse theorist *could* replace the traditional set-theoretic non-three colorability explanation with something of the form below (where the first ellipsis cover a first order logical claim about how the physical map regions are related by adjacency, and the second some specific first-order logical deduction).

> 'That map will never be three colored because it contains countries related by adjacency like ...., hence ... the map isn't three colored'

But this approach faces two problems. First, the explanations it produces are significantly less unifying and explanatory than the original explanation. In [11] Putnam famously contrasted unifying high-level explanations like 'this can't fit through that, because this is a square peg with side length ... and that is a round hole with diameter...' with the corresponding microphysical explanation one might give for the same fact. And (in the traditional explanatory indispensability literature) nominalistic paraphrases are commonly criticized for creating this kind of loss of unifying-explanatory power[16]. Replacing our non-three colorability explanation with a proof from specific adjacency facts

[16]See, for example, the criticisms Hartry Field's proposed infinitary nominalistic paraphrase of Newtonian mechanics in [4] in works like [3].

about the map we are considering which imply that it's not three colored, seems to involve such a loss.

Second, this strategy for dispensing with traditional set theory isn't sufficiently widely applicable. For it seems that we can entertain the conjecture (and perhaps even know) that some map is not three colored because it is not three colorable while *not knowing* specific facts about that map which entail it's not three colored. Perhaps we would need to know such facts to *prove* that the map isn't three-colorable. But, as recent work like [2] notes, we can sometimes explain physical facts by appealing to a mathematical claim whose truth we rationally suspect but haven't proved[17]. So it seems like a problem that the strategy for defending multiverse theory currently under consideration won't let us render such physical explanations.

To fix both problems, we could instead semantically ascend and talk about *derivability* in our scientific explanation, rather than giving a specific first-order logical deduction. We might say:

> The map isn't ever three colored because it's (first order logically) derivable from *some* true sentence about how finitely many countries are related by adjacency that it won't be three colored.

However, I will argue that this approach is also difficult to combine with Hamkins' view, or multiverse theory generally.

This problem for Hamkins using this approach is that there are prima facie strong links between the intuitive concepts of provability (which this strategy appeals to, without any caveat or explanation) and the intended natural number structure (which Hamkins rejects).

For example, Hamkins motivates skepticism about whether we can refer

[17]For example, Baker notes that scientists correctly hypothesized that bees had hexagonal honeycombs because this allowed for an optimal of side-length-to-area (under certain constraints) before they had any proof of the relevant mathematical fact (i.e., at a time when this was only a plausible/motivated conjecture).

to a unique intended natural number structure via skepticism about whether thoughts like '0, 1, 2 and so on' can secure a definite structure/stopping point for the natural numbers[18]. But this worry would seem to equally apply to our grasp of how many stages of inference a proof can contain[19][20].

Additionally (though this is not necessarily a problem) Hamkins may not be able to formalize/explain/express claims about provability via set theory in the way that traditional approaches can. For, from a traditional point of view, a claim is FOL provable iff there's a number Gödel coding a proof in the intended model of number theory. But Hamkins holds that every universe's copy of the natural numbers is non-standard from the point of view of some larger universe. So (if he accepted an ordinary concept of provability, as per this strategy) Hamkins might not be able to cash this notion out mathematically, in the way that traditional universe set theorists can.

Note, the issue here isn't that some set-theoretic universes could contain fake *three coloring functions* (presumably they cannot). Rather, it's that some set-theoretic universes will contain numbers corresponding to fake *proofs* of non-three coloring from some collection of truths about adjacency relations on the map. So, we get the following situation. In fact, whenever a map isn't three-colorable, we can (from a traditional realist point of view) derive the fact that it won't be three-colored from finitely many truths about adjacency relations on the map in FOL. But it's not clear that there's any claim that one can make

[18]See, for example, [7]

[19]We think proofs can have any finite number of steps, but you can't have infinite descending chains of proof steps (corresponding to non-standard models of number theory)

[20]Relatedly, we traditionally expect that a claim is provable in some formal system iff a number Gödel coding such a proof exists. But (in addition to rejecting a unique intended natural number structure), Hamkins seems to countenance universes that disagree on the truth value of such arithmetical provability claims. So there's a prima facie worry about whether universes which get provability facts 'wrong' (e.g., making $\neg Con(A)$ claims true, in cases where contradiction is not derivable from A) qualify as wrong for reasons unrelated to mathematicians' choice of which axioms to work with (contra Hamkins). Perhaps he could reply by saying that expected applications of set theory to provability are irrelevant to pure mathematics, like traditionally expected applications of geometry are irrelevant to physics?

about Hamkins' multiverse which expresses this provability claim.

What about other width multiverse theorists? Multiverse theorists can avoid all the specific worries for Hamkins just discussed by accepting that there's a unique intended natural number structure. However, even this move is not entirely without costs[21].

In any case, as noted above, I think the strategy of replacing set theoretic claims with provability claims doesn't answer the lazy explanatory indispensability worry for multiverse theorists, because it doesn't generalize in any obvious way to handle different kinds of seemingly cogent physical explanatory hypotheses.

6 Conclusion

In this paper, I've presented an explanatory indispensability worry for Hamkins' multiverse theory, (and multiverse approaches to set theory in general). I then suggested a few different strategies for answering this worry, and noted some problems for each.

In doing this, I don't claim to have refuted any version of the multiverse theory. Instead, I've tried to show how accepting multiverse set theory raises an immediate questions about what to say about logical possibility/the notion of 'all possible ways of choosing' and seeming appeals to this notion in applied mathematics. I think that in clarifying their favored answer this question, multiverse theorists like Hamkins would helpfully clarify their views about

[21] For example, Hamkins uses the general fact that different set universes can disagree on their ordinals to explain why we can't use Fregean abstraction to recursively introduce whole multiverse by (in effect) considering equivalence classes under a relation $\sim$ which relates two sets x and y occurring at a given level α in different universes (as intuitively playing the same structural role) iff it pairs up the elements of these two sets (c.f. Martin [8]). So perhaps the multiverse theorists who takes all universes to agree on their natural number structure faces a dilemma. If they reject Hamkins claim that universes disagree about the ordinals, they will need to find some other explanation for our inability to use Fregean abstraction principles to define a largest set theoretic universe in this way. On the other hand if they say that all universes agree on the intended model of the natural numbers but disagree on ordinals at higher stages, this can seem unprincipled.

pure mathematics as well [22].

A Hamkins' Multiverse

A.1 The Multiverse

In [6] Hamkins describes his multiverse proposal as a form of Platonism, which accepts the existence of many different set theoretic hierarchies (with equal mathematical status) rather than one unique intended hierarchy of sets. On Hamkins' view certain set theoretic statements like the Continuum Hypothesis (i.e., the claim that there is no set intermediate in size between the real numbers and the natural numbers) are not true or false simpliciter, but merely true in some parts of the multiverse and false in others. In some universes in the multiverse CH is true and in others it is false, and there is no unique intended universe. Thus (as Hamkins vividly explains in the passage below) CH cannot be settled by finding intuitively compelling new axioms from which it can be proved or refuted. For mathematicians' experience reveals there are parts of the multiverse in which CH holds and parts in which $\neg CH$ holds.

> "[If some obviously true seeming mathematical axiom] ϕ were proved to imply CH, then we would not accept it as obviously true, since this would negate our experiences in the worlds having ¬ CH. The situation would be like having a purported 'obviously true' principle that implied that midtown Manhattan doesn't exist. But I know it exists. I live there. Please come visit! Similarly, both the CH and ¬CH worlds in which we have lived and worked

[22] For example, I haven't discussed the possibility of supplementing the official ontology and ideology of Hamkins' Platonist multiverse with an appeal to primitive modal notions (of logical possibility), when cashing out physical explanations like the story about three colorabiltity above. I don't discuss this option because it's sufficiently different from the philosophical position Hamkins takes in [5]. However I think that some such modality-centric approach to set theory is ultimately the way to go, and I discuss how Hamkins could adopt a version of it in REDACTED.

seem perfectly legitimate and fully set-theoretic to us, and because of this, any [proof from ϕ that CH or that $\neg$CH] casts doubt on the naturality of ϕ. [6]

Three further features of Hamkins' multiverse are worth noting here.

First, (I take it) Hamkins isn't proposing any kind of supervaluationist theory on which ordinary set theoretic claims are determinately iff true in every set theoretic universe and determinately false iff in every such universe (so the facts above show that CH is *indeterminate*). The idea is that set theorists study different set theoretic universes in different contexts (as well as studying the relationships between them), like historians study different cities on earth. We don't say that it's indeterminate whether 'the city' has a population larger than 4 million, but rather by saying that there are many different cities, some of which have and others of which lack this property, and we must evaluate a historian's claim by determining which city they are talking about in a given context.

Second, Hamkins' proposal is inspired by a controversial interpretation of a mathematical technique called forcing. Hamkins suggests that for each set theoretic hierarchy V satisfying the ZFC axioms, we should accept that there is another (strictly wider) set theoretic hierarchy $V[G]$, the forcing extension of V. This expanded universe $V[G]$ adds a set G to V, where G is subset of a set (a partial order $\mathbb{P}$) that's already in V — along with other sets, as needed for $V[G]$ to satisfy the ZFC axioms.

A version of this claim is uncontroversially true; if we work in some background notion of set theory we can prove that that every *countable model* of the ZFC axioms for set theory has a forcing extension (as mainstream/conventional approaches to forcing do)[23]. In contrast, Hamkins endorses the general claim

[23] I omit mention of other technical devices Boolean valued models and inner models, for understanding forcing arguments without admitting that our background set-theoretic universe could

that every set-theoretic hierarchy satisfying the ZFC axioms has a forcing extension. He thus contradicts traditional/mainstream views that the intended hierarchy of sets already contains 'all possible' subsets all sets it contains, – so there can be no extended universe V[G], which adds a missing subset to a set (the partial order $\mathbb{P}$) this V already contains.

Finally, Hamkins asserts more powerful principles than the above claim about taking forcing extensions. Specifically, he makes the provocative claim that, "Every universe V is ill-founded from the perspective of another, better universe."[5][24] (while no process of repeatedly taking forcing extensions can generate such a universe). Note that, in such cases, the natural numbers in V will be non-standard from the perspective of V', meaning that different set-theoretic hierarchies can have different views about what number theoretic claims are true.

B Generalizing the Argument of §4 to Target the Weak Multiverse Theorist

In §4 I argued that Hamkins can't answer our lazy explanatory indispensability worry by claiming the multiverse as a whole contains sets corresponding to all possible ways of choosing. In this appendix I'll develop an anlogous argument that multiverse theorists in general can't answer the lazy explanatory indispensability worries in this way.

I will do this by constructing a sentence which – given the assumption that the multiverse as a whole contains sets witnessing all possible ways of

literally be widened in the way described above.

[24] To explain this talk of one set-theoretic universe being well founded from the perspective of another, note that a model of ZFC set theory is well founded iff its ordinals are well founded. In particular, a larger model of ZFC set theory V′ can see a smaller model of set theory inside it, V, as not being well founded, because V′ may contain a subset of one of the ordinals o in V′ that's missing from V′ and which doesn't have an ∈ least member.

choosing from physical objects (as per §4) – is true iff the continuum hypothesis (as intuitively understood) is false. This claim will, in effect, say that the stars have the cardinality of the (informally intended) natural number structure, the pebbles have the cardinality of the (informally intended) powerset of the natural number structure, and the red pebbles have a cardinality strictly in between that of the stars and the pebbles. Accordingly, it will be true iff there is some possible way of choosing some objects xx from among (objects with the intended structure) the natural numbers, which have a cardinality strictly between $\aleph_0$ (that of the natural numbers) and $2^{\aleph_0}$ (that of the real numbers).

We've already seen how to say (by exploiting the assumptions that there are sets somewhere in the multiverse witnessing all possible ways of choosing physical objects) that the stars form an intended model of the Peano Axioms – and hence have cardinality $\aleph_0$.

How how do we say something which implies there are pebbles witnessing all possible ways of choosing some of these stars (and hence have the cardinality we informally expect from $2^{\aleph_0}$)? Plausibly we could have pebbles that have a robust disposition to glow or not glow when brought near each star.

- Each pair of pebbles is disposed to in when brought near a distinctive of stars[25].
- For any *set* of stars anywhere in the multiverse, there is a pebble that glows when near exactly those stars.

Given the assumption that there are sets (somewhere in the multiverse) corresponding to all possible ways of choosing some stars, this pair of claims ensures that the pebbles have the cardinality of the powerset of the stars. Thus we can write down a description of the pebbles and stars which intuitively implies that

[25] That is, for each pair of pebbles x and y, if x is not identical to y, there is some star z such that x is disposed to glow when brought near z and y is not, or vice versa.

the stars have cardinality $\aleph_0$ and the pebbles have cardinality $2^{\aleph_0}$.

Furthermore, we can write a claim which intuitively requires that the red pebbles have cardinality strictly between that of the stars and that of all the pebbles. For remember that, by assumption, our multiverse theorist recognizes that (for any metaphysically possible world w) and any possible way of choosing some physical objects (or pairing them up with some relation), there's a corresponding set (or set of sets with ur-elements coding a corresponding relation) *somewhere* in the multiverse. This is what allows them to claim that they can render all the same kinds of physical-combinatorial explanatory hypotheses (like the three-coloring explanation above) the conventional universe theorist can, while embracing multiverse theory. So it suffices to conjoin the following claims.

- There's a set (somewhere in the multiverse) coding a function from the stars to the red pebbles (i.e., a possible way a relation could relate stars to pebbles), and no set (anywhere in the multiverse) coding a 1-1 onto function from the red pebbles to the stars. (So, intuitively, there are strictly more red pebbles than stars.
- There's no set (anywhere in the multiverse) coding a 1-1 onto function from all the pebbles to the red pebbles. (So, intuitively, there are strictly more pebbles total than red pebbles.)

Now I claim that CH is intuitively false if it would be metaphysically possible to have this composite description be true, while some objects – say, the red pebbles – had cardinality strictly in between that of the pebbles and the stars (in the sense expressed above). And (more interesting) CH is intuitively true if it would not be possible to have the red pebbles as above. I take the metaphysical possibility of the truth of the above descriptions of pebbles and stars to be clear —- or to become so if my talk of pebbles is replaced by less picturesque talk of point particles[26]. And (by a kind of Humean recombination intuition) I take it that if there *were some possible way of choosing* some pebbles with cardinality between that of the stars and the pebbles (as required for CH to be false), then would be metaphysically possible for the facts about which pebbles are red to witness this possible way of choosing.

Thus, I claim that combining the descriptions of the stars, pebbles and red pebbles above yields a sentence which is intuitively true if and only if CH is false (given our target multiverse theorist's assumption that all possible ways of choosing some physical objects/n-tupples of physical objects are witnessed by the existence of a set somewhere in the multiverse).

I realize that some multiverse theorists may be willing to give up some of the assumptions about metaphysical possibility employed in the argument above (or even give up the notion of metaphysical possibility all together). My aim in this paper is not to vanquish width multiverse theory and sow the fields with salt, but rather to highlight how different versions of multiverse theory (that answer my explanatory indispensability challenge differently) can have substantive and interestingly different philosophical commitments.

References

[1] A. Baker. Are there Genuine Mathematical Explanations of Physical Phenomena? *Mind*, 2005.

[26] Note that the kind of multiverse theorists currently under consideration can't say that general cardinality considerations prevent the simultaneous existence of this many pebbles (as an 'everything is countable' view which doesn't take there to be a complete Platonist multiverse of universes whose sets we can simultaneously quantify over). For they claim that in this situation there would (and our quantifiers would range over) *sets* in the multiverse as a whole witnessing all possible ways of choosing from ur-elements. So they are committed to the simultaneous existence (and our ability to quantify over) sets as plentiful as we are now supposing the pebbles to be.

[2] Alan Baker. Mathematical Accidents and the End of Explanation. In Otávio Bueno and Øystein Linnebo, editors, *New Waves in Philosophy of Mathematics*, pages 137–159. Palgrave Macmillan UK, London, 2009.

[3] Mark Colyvan. *The Indispensability of Mathematics*. Oxford University Press, 2001.

[4] Hartry Field. *Science Without Numbers: A Defense of Nominalism*. Princeton University Press, 1980.

[5] Joel Hamkins. The multiverse perspective in set theory, August 2013.

[6] Joel David Hamkins. The Set-Theoretic Multiverse. *Review of Symbolic Logic*, 5(3):416–449, 2012.

[7] Joel David Hamkins. Set theoretical multiverse and truths, July 2023.

[8] Donald A. Martin. Multiple Universes of Sets and Indeterminate Truth Values. *Topoi*, 20(1):5–16, 2001.

[9] Vann McGee. How We Learn Mathematical Language. *Philosophical Review*, 106(1):35–68, 1997.

[10] Hilary Putnam. *Philosophy of Logic*. London: Allen & Unwin, 1971.

[11] Hilary Putnam. Philosophy and Our Mental Life. In *Mind, Language, and Reality*. Cambridge University Press, 1975.

[12] W. V. Quine. On What There Is. In W. V. Quine, editor, *From a Logical Point of View*, pages 1–19. Cambridge, Mass.: Harvard University Press, 1961.