# TAKING THE ANALOGY BETWEEN SET THEORY AND GEOMETRY SERIOUSLY

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Abstract. Set theorist Joel Hamkins uses considerations about forcing arguments in set theory together with an analogy between set theory and geometry to motivate his influential set-theoretic multiverse program: a distinctive and powerfully truthvalue-antirealist form of plenetudinous Platonism. However, I'll argue that taking this analogy seriously cuts against Hamkins' proposal in one important way. I'll then note that putting a (hyperintensional) modal twist on Hamkins' Multiverse lets take Hamkins' analogy further and address his motivations equally well (or better) while maintaining naive realism.

## 1. Introduction

Philosophers who want to deny that there are right answers to mathematical questions (like the Continuum Hypothesis) which are undecidable via deduction from mathematical axioms we currently accept often draw on an analogy between set theory and geometry [6, 12][1]. In this paper I will discuss one of the most important and interesting recent examples of this phenomenon: set theorist Joel Hamkins' use of (a specific form of) this analogy to motivate his set-theoretic multiverse program.

Hamkins has developed an influential[2] multiverse approach to set theory, on which there are many different hierarchies of sets, and there's no fact of the matter about whether certain set-theoretic statements are true, beyond the fact that they are true of some hierarchies of sets within the multiverse and false in others. On this view there is no 'full' intended hierarchy of sets which contains all subsets of sets it contains- or even all subsets of the natural numbers. Rather, for *every* set theoretic

[1]Strictly speaking, in [12] Maddy advocates a pragmatist attitude to axiom choice not an antirealist one, but the same considerations apply.
[2]See, for example, [**?**]

universe $V$ in the multiverse, there's an extending 'fatter' hierarchy of sets $V[G]$ that includes all sets in $V$ but but also an extra 'missing subset' of the set of natural numbers in $V$.

In this paper, I'll discuss the two motivations Hamkins gives for this project in [8, 9]: an appeal to phenomenology and analogy between set theory and geometry. Hamkins' view is complex, and I don't claim to refute it here. However, I will argue that there's an important limit to how much Hamkins' analogy between set theory and geometry can be used to support his current plenetudinous Platonist formulations of the multiverse proposal.

In the case of geometry we have two things: a range of different geometries which constitute equally legitimate objects for (non-formalist) mathematical study *and* (fairly) determinate facts about physical geometry in our universe. Hamkins' multiverse proposal nicely mirrors the former idea, but provides nothing corresponding to the latter. Indeed I'll argue that the key controversial feature of Hamkins' multiverse prevents us from telling a natural and attractive story about what could make something the correct set theory for our universe.

This creates a problem because it means that the change in attitudes to set theory Hamkins advocates is more radical than the change in attitudes to geometry he invokes for motivation. In the case of geometry, we say there's an important joint-carving notion (physical geometry), that can take over the scientific-explanatory work done by appeals to naive geometry and explain the attraction of naive geometry. In contrast, Hamkins seems forced to say that no legitimate notion was got at by analogous seemingly explanatory appeals to a naive notion of 'all possible ways of choosing' from a given collection.

Indeed I'll argue that a truth-value realist about set theory can equally well address Hamkins' stated motivations (both appeals to forcing phenomenology and analogies to geometry) by putting a (hyper-intensional) modal twist on the multiverse. Doing this lets us mirror both aspects of our attitude to geometry noted above.

Accordingly, if my argument succeeds, it suggests that, while Hamkins' considerations may teach us something deep about the philosophy of set theory, they don't do much to motivate his truth-value anti-realism.

## 2. Hamkins' Multiverse

2.1. **Forcing Fundamentals.** Before describing Hamkins' multiverse, I will first review some very basic mathematical facts about forcing, as this technique plays a central role in Hamkins' program (and some of these mathematical facts will play an important role in my argument).[3] In particular Hamkins' main motivation for the Multiverse, aside from the analogy with geometry, arises from the idea that we should understand mathematical arguments by forcing in a certain unconventional way.

Set-theoretic forcing was, famously, developed by Paul Cohen to prove the independence of the Continuum Hypothesis (i.e., the claim that there is no set intermediate in size between the real numbers and the natural numbers). However this method has been generalized to prove a broad range of meta-mathematical results.

As standardly presented, forcing is a technique which lets one produce a new model of set theory from an original *countable* well-founded[4] model $M$ of set theory.

We work in the total hierarchy of sets $V$, using assumptions like the standard ZFC (Zermelo-Frankael plus Choice) axioms of set theory. And consider an infinite partial order $\mathbb{P}$ that is a set in our countable model $M$. Because $M$ is countable, it has to be missing some subsets of any infinite set $\mathbb{P}$ it contains, by Cantor's diagonal argument. Our strategy will be to expand $M$ by adding a missing subset of this set $\mathbb{P}$[5] to $M$.

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[3]Many thanks to REDACTED for help with this section, and thanks to REDACTED for much relevant lecture and informal conversation.

[4]More specifically, forcing lets you produce a new model of set theory extending every countable *transitive* model of set theory. A model $M$ of set theory is transitive iff the membership relation in $M$ is $\in$, i.e. $x \in_M y \leftrightarrow x \in y$. However, by the Mostowski collapse lemma [11], any well-founded countable model is isomorphic to a transitive model.

[5](In the famous originating case P was the set of functions from $\omega$ to 1,0 which decide which..)]

Specifically, we can use the fact that $M$ is countable to prove that there's an '$M$-Generic' set $G \subset \mathbb{P}$, (where being $M$-generic implies *not* a being set in $M$)[6].

Next we consider a fatter model of set theory $M[G]$ which expands $M$ by adding $G$ to it (along with other sets, as needed to satisfy the ZFC axioms[7]. And finally we show that any such $M[G]$ must satisfy some desired claim $\phi$. In this way we prove the relative consistency of ZFC $+\ \phi$.

But now the key point about forcing arguments that opens the door to Hamkins' multiverse (and the reason it is called the multiverse) is this. The mechanics of forcing allow us to make claims that *only quantify over* sets in original countable model of set theory[8] $M$ but can be seen as *implicitly telling us about* this larger model of set theory $M[G]$[9] in the following sense.

We can define a relation $\Vdash$ (called a **forcing relation**) such that the claim that $\Vdash \phi$ only involves sets in $M$ but we can prove the following biconditional (without appeal to the fact that $M$ is countable). If there is any $M$-generic subset of $\mathbb{P}$:

$\Vdash \phi$ if and only if $M[G] \models \phi$ for every generic $G \subset \mathbb{P}$.

That is, a sentence $\phi$ is forced ($\Vdash \phi$) if and only if for any generic set G of the kind mentioned above, $\phi$ is true in the expanded model of ZFC $M[G]$ we get by adding $G$.

A specific forcing argument proceeds by picking an infinite partial order $\mathbb{P}$ which we will add a subset of, and then proving that $\Vdash \phi$ holds when $\phi$ is some claim we wish to show is consistent with the ZFC axioms.

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[6]Specific, a generic, i.e., a generic filter $G$ is a filter which intersects every dense subset of $\mathbb{P}$ included in $M$.

[7]$M[G]$ winds up being the smallest transitive model of ZFC extending $M$ and containing $G$ as a set.

[8]i.e. model of ZFC

[9]While $M$ can't define truth in $M[G]$ in $M$ one can define a class of names for objects in $M[G]$ (some of which may refer to the same object) and a forcing relation $p \Vdash \phi$ (where $\phi$ is a sentence in the language of set theory and $p$ an element of the forcing partial order $\mathbb{P}$ supplemented with the aforementioned class of names) which holds just if $M[G] \models \phi$ for every generic object $G$ containing $p$.

So for instance, Cohen proved in ZFC that there is a partial order $\mathbb{P}$ such that $\Vdash \neg$CH (where CH is the continuum hypothesis). Thus, if $M$ is a *countable* transitive model of ZFC then (if $G$ is a generic object for $\mathbb{P}$), $M[G]$ is a countable transitive model of ZFC+$\neg$CH in $M[G]$. Of course, speaking formally, we can't assume that there are *any* models of ZFC but this is enough to establish the consistency of ZFC+$\neg$CH provided we think ZFC is consistent (and hence has a countable model).

2.2. **Hamkins' Proposal.** Hamkins describes his multiverse proposal as a form of Platonism.

> The multiverse view is one of higher-order realism—Platonism about universes— and I defend it as a realist position asserting actual existence of the alternative set-theoretic universes into which our mathematical tools have allowed us to glimpse. [9]

However, rather than accepting a single platonic hierarchy of sets, he proposes that there are many different hierarchies of sets. The set-theoretic multiverse is the space of all such set-theoretic hierarchies. And certain set-theoretic statements, like the Continuum Hypothesis are not true or false simpliciter, but merely true in some parts of the multiverse and false in others.

As Hamkins vividly explains in the passage below, $CH$ cannot be settled by finding intuitively compelling new axioms from which it can be proved or refuted, because mathematicians' experience reveals there are parts of the multiverse in which $CH$ holds and parts in which $\neg CH$ holds.

> "[If some obviously true seeming mathematical axiom] $\phi$ were proved to imply $CH$, then we would not accept it as obviously true, since this would negate our experiences in the worlds having $\neg CH$. The situation would be like having a purported 'obviously true' principle that implied that midtown Manhattan doesn't exist. But I know it exists; I live there. Please come visit! Similarly, both the

> CH and $\neg CH$ worlds in which we have lived and worked seem perfectly legitimate and fully set-theoretic to us, and because of this, any proof from $\phi$ that CH or that $\neg CH$ casts doubt to us on the naturality of $\phi$ [9].

Hamkins' view of the multiverse is heavily influenced by the set-theoretic technique of forcing just described. In particular, he suggests that for any set-theoretic hierarchy $V$ we should accept that (for an appropriate partial order $\mathbb{P}$ in $V$) there is another set-theoretic hierarchy $V[G]$ corresponding to the forcing extension of $V$ with respect to the partial order $\mathbb{P}$. As we saw in the previous section, this claim is straightforwardly true if we work in some background notion of set theory and take $V$ to be a *countable* model of set theory. But Hamkins suggests we adopt it more generally for any set-theoretic hierarchy. Specifically, he contends that

> A stubborn geometer might insist—like an exotic-travelogue writer who never actually ventures west of seventh avenue—that only Euclidean geometry is real and that all the various non-Euclidean geometries are merely curious simulations within it. Such a position is self-consistent, although stifling, for it appears to miss out on the geometrical insights that can arise from the other modes of reasoning. Similarly, a set theorist with the universe view can insist on an absolute background universe $V$, regarding all forcing extensions and other models as curious complex simulations within it. (I have personally witnessed the necessary contortions for class forcing.) Such a perspective may be entirely self-consistent, and I am not arguing that the universe view is incoherent, but rather, my point is that if one regards all outer models of the universe as merely simulated inside it via complex formalisms, one may miss out on insights that could arise from the simpler philosophical attitude taking them as fully real [9].

Now this claim that we can extend *every* set-theoretic structure by taking a forcing extension is an interesting and controversial aspect of his view. For note that it directly conflicts with the standard realist intuition that it's possible to build a set-theoretic hierarchy that already contains 'all possible subsets' of any set in that hierarchy. For any such set-theoretic hierarchy $V$ must already contain all subsets of every partial order $\mathbb{P}$ it contains. Thus, there should not be any generic $G \subset \mathbb{P}$ which isn't a member of $V$, i.e., $V$ and $V[G]$ should always be the same. For instance, if one thinks that a set-theoretic hierarchy already contains all possible subsets of the integers, it would be impossible to extend that hierarchy via a forcing extension which adds another subset of the integers.

While Hamkins' proposal seems to take significant motivation from the example of forcing extensions, this isn't the only closure principle about the multiverse which he accepts. It isn't even the most controversial. He also suggests that every set-theoretic hierarchy $V$ is countable from the perspective of some other hierarchy $V'$ [9]. Indeed, he suggests that - although "this principle appears to be abhorrent to most set theorists" - every set-theoretic hierarchy $V$ is ill-founded from the 'perspective' of another set-theoretic hierarchy $V'$.

## 3. Motivating the Multiverse

Why should one accept this radical approach to set theory? In this paper, I'll discuss two motivations Hamkins gives in his philosophical overview 'The Multiverse Perspective in Set Theory' [8] and suggest that we can more attractively accommodate these motivations by giving a hyperintensional modal twist to Hamkins' existing Plenitudinous Platonist formulation of his multiverse program.

3.1. **Mathematical Practice and Phenomenology.** First, Hamkins appeals to the practice and phenomenology of set theory. He notes that now, rather than stating results proved by forcing as consistency claims of the form $Con(\text{ZFC}+\phi) \rightarrow Con(\text{ZFC}+\psi)$, "contemporary work would state the theorem as: If $\phi$, then there is a forcing extension that satisfies $\psi$." The latter claim could either be read as asserting the existence of a forcing extension of your total $V$ rather than any countable model satisfying ZFC $+ \psi$. Hamkins' Multiverse hypothesis takes this appearance at face value.

Hamkins also appeals to the phenomenology of making forcing arguments, which he describes as follows and claims that forcing takes at face value.

> [The multiverse proposal] makes sense of our experience—in a way that the universe view does not—simply by filling in the gaps, by positing as a philosophical claim the actual existence of the generic objects which forcing comes so close to grasping, without actually grasping. With forcing, we seem to have discovered the existence of other mathematical universes, **outside our own universe**, and the multiverse view asserts that yes, indeed, this is the case. We have access to these extensions via names and the forcing relation, even though this access is imperfect. Like Galileo, peering through his telescope at the moons of Jupiter and inferring the existence of other worlds, catching a glimpse of what it would be like to live on them, set theorists have seen via forcing that divergent concepts of set lead to **new set-theoretic worlds, extending our previous universe**, and many are now busy studying what it would be like to live in them. [8] pg. 11

Equally eminent set theorists who reject the multiverse program [13] might give a different description of the phenomenology. And even if one grants this point, one it's disputable whether the multiverse better fits mathematical practice and phenomenology than conventional realist approach to set theory (paired with the conventional interpretation of forcing described above). Admittedly the conventional set theorist can't account for the bolded part (emphasis mine) in Hamkins' description, about our seeming access to a universe genuinely extending the one we're currently working in. However, one might argue that traditional realist approaches to set theory account for many more aspects of mathematical intuition and practice overall than Hamkins' theory does. For (as we saw above) Hamkins admits that his own principles about what hierarchies exist in the multiverse will be "abhorrent to many set theorists."

However, Hamkins has a second way of motivating the multiverse, which may have more power to show a clear advantage of the multiverse perspective over more traditional realism: the analogy between set theory and geometry. This will be my main target in this paper [9].

3.2. **An Analogy Between Set Theory and Geometry.** I will quote Hamkins' way of laying out the analogy between set theory and geometry at some length, because it presents the main target to be attacked in this paper. He writes,

> There is a very strong analogy between the multiverse view in set theory and the most commonly held views about the nature of geometry. For two thousand years, mathematicians studied geometry, proving theorems about and making constructions in what seemed to be the unique background geometrical universe. In the late nineteenth century, however, geometers were shocked to discover non-Euclidean geometries. At first, these alternative geometries were presented merely as simulations within Euclidean geometry, as a kind of playful or temporary re-interpretation of the basic geometric concepts. For example, by temporarily regarding 'line' to mean a great circle on the unit sphere, one arrives at spherical geometry, where all lines intersect; by next regarding 'line' to mean a circle perpendicular to the unit circle, one arrives at one of the hyperbolic geometries, where there are many parallels to a given line through a given point. At first, these alternative geometries were considered as curiosities, useful perhaps for independence results, for with

> them one can prove that the parallel postulate is not provable from the other axioms. In time, however, geometers gained experience in the alternative geometries, developing intuitions about what it is like to live in them, and gradually they accepted the alternatives as geometrically meaningful. Today, geometers have a deep understanding of the alternative geometries, which are regarded as fully real and geometrical [8].

In this quote, Hamkins compares set theorists who approach forcing conventionally (as studying countable models inside the true intended hierarchy of sets V) to old geometers who took studying non-euclidean geometries to be legitimate mathematics but only to reveal syntactic facts about provability and consistency, plus what would be true under some "playful reinterpreations" of the terms "point" and 'line" in these axioms. He suggests that set theorists should mirror the step we took in geometry to regarding alternate axiom systems as "geometrically meaningful" and "alternate geometries ... as fully real" and that adopting the Multiverse theory corresponds to doing this.

Accordingly, to evaluate the strength of this parallel and the power of Hamkins' motivation by analogy, we'll need to get a grip on how Hamkins is thinking about the transition in our attitudes to geometry. I think this is especially important because what Hamkins describes is rather different from what may first spring to mind.

In the stage corresponding to contemporary mainstream set theory and understandings of forcing, Hamkins writes that:

> At first, these alternative geometries were presented merely as simulations within Euclidean geometry, as a kind of playful or temporary re-interpretation of the basic geometric concepts. For example, by temporarily regarding 'line' to mean a great circle on the unit sphere, one arrives at spherical geometry, where all lines intersect;

> by next regarding 'line' to mean a circle perpendicular to the unit circle, one arrives at one of the hyperbolic geometries, where there are many parallels to a given line through a given point....[T]hese alternative geometries were considered as curiosities, useful perhaps for independence results, for with them one can prove that the parallel postulate is not provable from the other axioms.

But, in time, he says:

> In time, however, geometers gained experience in the alternative geometries, developing intuitions about what it is like to live in them, and gradually they accepted the alternatives as geometrically meaningful.

Note that Hamkins describes the process of coming to see alternatives as getting at something real and geometrical in terms of **building up** a new way of thinking about alternate axiom systems rather than what may be more familiar: **somehow debunking or rejection** the expected connections between geometrical claims and claims about physical space.

A common and more formalist way of thinking about the adoption of modern attitudes towards geometry involves unshackling mathematical geometry from any intended applications. The mathematician studies a priori what various axioms about points and lines imply. On this view, the scientist may later state an empirically motivated theory involving bridge laws which say that a certain collection of geometrical axioms hold if we interpret 'line' to mean physical line. But, any such physical applications or imagery are irrelevant to mathematics. Accordingly, our impression that considering great circles on a sphere in a euclidean space was a mere toy model or playful reinterpretation turned out to be an illusion. The interpretation of line as meaning great circle within (what we would have naively thought of as) a Euclidean space was no more or less a reinterpretation or mere toy model than any other physical interpretation of the world 'line' as it occurs in geometrical axioms.

But, this is not the change in attitudes to geometry which Hamkins invokes and wants us to mirror in the case of set theory. For one thing, the FOL axiom transcendent notion of containing 'all possible subsets' doesn't seem to be particularly physical (rather than mathematical), so learning to separate intended physical applications from mathematics doesn't cut against the conventional point of view. Nor does Hamkins suggest any reason *antecedent to the argument he's currently trying to make* for thinking the notion of containing all possible subsets is incoherent (as one might perhaps argue that naive conceptions of what picked out the true geometry were).

Finally, and more abstractly, note that merely coming to regard all models of certain set theoretic axioms as equally intended (i.e., discarding our naive ZFC transcendent expectations that seemed to pick out a correct interpretations of 'set' as mathematically irrelevant) can't motivate the revolution in attitudes to forcing Hamkins is arguing for. Someone who starts from conventional realism and takes *this* moral away from the history of geometry will agree with Hamkins in saying that there's no right answer to CH (as they now regard countable models of ZFC+CH and ZFC+¬CH as equally intended. But they won't have any reason to say that for each model there's a fatter one. They will merely regard the fact that one of these structures is the fattest as just an interesting property that one hierarchy of sets happens to have (analogous to all finite groups being analyzable in terms of simple groups). And they will reject any suggestion that consideration of alternate set theories via forcing gets us at something ("full grown set theoretic universes" [8]) we're more interested in than the countable models presented by the conventional perspective.

Instead of any such scruitny and debunking Hamkins describes a process of *positively working* with different geometries that only seem to have toy models and getting a sense of "what it's like to live in them." So his language suggests a process of coming to see these variant axioms as true of a structure that (in some important sense) isn't a mere toy model. What could does this involve?

Further remarks where Hamkins seems to endorse the possibility of literally "liv[ing] in" (as opposed to merely mathematically *working in*) some variant axiom system are highly suggestive. He seems to distinguish three different states: proving facts 'about' $V[G]$ while working in $V$ (via forcing), "jumping in" to $V[G]$ by working in it (i.e. reasoning from axioms that truely describe it), and thirdly (!) actually living in $V[G]$[10]. For when mathematicians do the second thing (by working in $V$), he sometimes describes them as merely "reasoning *as though*" they were doing the third thing (i.e., living in $V$).

Accordingly, the following interpretation of what it means to come to see variant axiom systems as geometrically meaningful (in the sense Hamkins wants to use for his argument by analogy) seems to me very natural, if not inevitable.

Mathematicians come to see variant geometrical axioms as correctly describing the behavior of physical points and lines in a (conceivable) physical space[11] in contrast to merely true on some intuitively unacceptable sharpening of our concept 'physical line' which made us dismiss the example of the great sphere as a mere toy model[12].

[10]He writes that just as geometers can "in a sense", "jump inside the alternative geometry, for example by adopting particular negations of the parallel postulate and reasoning totally within that new geometrical system", set theorists can "reason about the forcing extension by jumping into it and reasoning as though they were living in that extension." So it seems that "jump[ing] inside" an alternative geometry/set theory means *reasoning within* some new axiom system. And it seems that this only counts as reasoning *as though* one were living in an alternative geometry/set theory.

[11]Here we might take physical lines to be understood in some folk 'manifest image' way or as tied to the definitions of physical lines used by recent scientific theories. Note, I don't think giving this interpretation depends on assuming that there's a unique intended notion of physical points and lines, but rather our sense that -for reasons I won't positively analyze here associating straight lines with paths of light is more appropriate than associating great circles on a sphere with lines- i.e. the former is a better choice of name for a physicist stating their theory, because the role of paths of light has a closer relationship to spatial lines in the manifest image or previous theories about physical lines. Much philosophy of science can and has been done on this topic, but which particular philosophy you use to cash out intuitive distinction Hamkins mentions - the sense Hamkins appeals to [**?**] in which Einstein showed that space is non-euclidean but considering circles on a great sphere doesn't - won't matter here.

[12]i.e. as merely involving truth under a reinterpretation

Accordingly, the following seems like a natural question to ask if we want to take this analogy seriously as advocating some attitude to set theory. What is involved in literally living in a world described by different set theoretic axioms? But, as we will see below, answering this question turns out to be quite tricky for the proponent of Hamkins' multiverse (at least as currently stated).

## 4. Taking The Analogy Seriously

In this section I'll argue that Hamkins Platonism about the multiverse almost forces him to allow an important disanalogy between 'naive' geometry and 'naive' set theory. If so, Hamkins can't motivate his view by saying that it's just what falls out from treating set theory and geometry the same way.

In the case of geometry, in addition to the study of variant geometries (like those Hamkins mentions in the quote above) there's a further question: what's the geometry of the physical space we live in?[13] The change of opinions about geometry alluded to above didn't deny the existence of robust metaphysically joint carving laws with the physical consequences naive geometry had claimed. It just downgraded these laws from metaphysical necessities to physical laws. Appeal to physical geometry provides an important sense in which, e.g., the parallel postulate is definitely false (which is not relative to a choice of axioms to work in).

Accordingly we can ask what geometry someone 'lives in' in two senses. We can ask (in the somewhat metaphorical sense invoked above) what axiom systems they're employing and studying. And we can also ask (more literally) what axioms describe the space their body physically occupies, i.e., the physical geometry of the universe we all share.

But in [8] Hamkins only explicitly develops the first (metaphorical) sense in which a person can 'live in' a set-theoretic universe satisfying certain axioms. For example, he suggests that set theorists like himself know that there are both CH and $\neg$CH

[13] That is, there are facts constraining the behavior of all actual spatial points and lines etc, as well as facts about what's possible within various alternate geometries we can metaphorically visit and imagine living in by doing mathematics with different axioms.

worlds within the multiverse, because they have "lived and worked" in such worlds by reasoning from the axioms ZFC+CH and ZFC+¬CH.

However, as noted above, Hamkins also seems to allow a more robust (contrasting) sense in which we could all be said to 'live in' a reality that satisfies certain set-theoretic axioms and not others. For example, in claiming that set theorists can "reason about a forcing extension by jumping into it and reasoning **as though** they were living in that extension." (emphasis mine), Hamkins seems open to a notion of actually living in a world corresponding to some $V[G]$ within the set-theoretic multiverse (as opposed to merely reasoning as if one did).

But what is this supposed to mean? What (if anything) could it mean for, say, the Continuum Hypothesis to be true of the set-theoretic structure of our reality in a sense analogous to the one in which the parallel postulate is false of the geometrical structure of our reality? No story is provided.

4.1. **Living In A Given Set-Theoretic Universe.** I'll will now argue that there's a very natural answer to the above question, but this answer is (unfortunately) immediately incompatible with Hamkins' current Platonistic development of the multiverse proposal.

I suggest that naive geometry attempts to study how it is (in some sense) possible for points and lines in space to be related. The naive iterative hierarchy conception of set theory attempts to study how it is (in some sense) possible to choose from an antecedently given plurality of objects. That is, it attempts to study general combinatorial constraints on how any predicate (definable with parameters) could apply to some of these objects.

When we naively appeal to a notion of the hierarchy of sets containing, at each successor stage, 'all possible subsets' of the sets formed below, we take there to be definite objective constraints on how it's possible to chose some objects from within any given plurality of objects. This idea that each layer of sets witnesses all possible ways of choosing some objects from previous layers is expressed in Boolos' (representative) characterization of the iterative hierarchy conception of sets in [5]. It also plays an important role in the way that we apply set theory with ur-elements. For example, we take the fact that there are only 8 distinct sets[14] whose elements are all bowls of sundae toppings on a certain table (i.e., only 8 possible ways of choosing which if any of these toppings to add), to predict and explains why there will never be 9 people who choose differently from these toppings. Similarly, we'd take a proof that there's no set-theoretic function from countries on some physical map with infinitely many countries to the set $\{1, 2, 3\}$ which three colors that map to predict and explain why that map will never actually be three colored (or three scented or etc).

So a natural thought is that current (and from Hamkins' point of view 'naive') iterative hierarchy set theory attempts to study general constraints on 'how it would be possible to chose some of' a given plurality of objects via studying a hierarchy of sets which gets these facts right. Such a hierarchy must contain sets corresponding to all possible ways of choosing elements from sets in the hierarchy. Hence, the set-theoretic hierarchy must (intuitively) satisfy all instances of the following comprehension schema (where the $\Box$ expresses metaphysical or logical necessity) and the quantifiers range over all objects, not just sets in $V_@$[15].

Necessary Comprehension

$$\Box \forall z \in V_@ \forall w_1 \forall w_2 \ldots \forall w_n \exists y \in V_@ \forall x [x \in y \Leftrightarrow ((x \in z) \land \phi)].$$

[14]Or, for those with less quantifier variantist/platonist inclinations, this Platonic fact about set existence reflects a deeper underlying modal fact about how it would be possible to choose elements of a set, which predicts and explains both the regularity in sets and the regularity in sundae choice at issue.

[15]Personally I'd say we expect this necessity because we take induction/fatness principles to reflect a deeper 'combinatorial' necessity (logical necessity given structural facts) which we can highlight by comparing induction on numbers and conception of width of sets to claims about which maps are three colorable, or how it's possible to traverse the Königsberg bridges, we accept first order schemas but take them to apply with special necessity and generality because we think the math objects have a certain property which 'combinatorially' ensures certain things. If the structure of the numbers/pure sets has this property and we expect this structure to be the same at all metaphysically and physically possible worlds then we should expect instances of induction/fatness to hold for all predicates and all relations, including predicates specified by appeal to other physical or mathematical structures.

Now what shall we do if we want to mirror the history of geometry which Hamkins invokes above?

I propose that the historical transition in attitudes to geometry which Hamkins invokes corresponds to (something like) a mere downgrading the kind of necessity attributed to certain a priori intended physical applications of geometry — from metaphysical to merely physically necessity. Naive geometry attempted to study elegant and joint-carving geometrical laws that implied metaphysically necessary constraints on the behavior of all points and lines. And, after the transition, this expectation of there being some elegant joint carving [16] geometrical laws (ensured by the structure of the physical space we live in) constraining the behavior of all actual physical points and lines to be discovered remained. So did naive expectations about how truths concerning this favored geometry imply constrains on the behavior of all *actual* physical points and lines were preserved (e.g. the expectation that if the parallel postulate is true in this favored geometry then there must not actually be any parallel physical lines that intersect). We just began to allow that it would be (metaphysically) *possible* for different geometrical laws to constrain the behavior of all physical points and lines—and thus to treat certain alternative axiom systems as legitimate objects of non-formalist 'genuinely geometrical' mathematical investigation (in the way Hamkins emphasizes).

But if one accepts this picture then what falls out of treating set theory like geometry can't possibly be Hamkins Platonistic multiverse. For the natural parallel to the above approach to geometry would seem to be the following view.

*Continue to accept* naive expectations that there are elegant metaphysically joint carving laws about 'all possible ways of choosing' from an antecedently given plurality of objects to be discovered. And continue to accept naive expectations that truths concerning this favored notion of 'all possible ways of choosing' constrain how all predicates actually apply. So, for example, allow that all instances of comprehension schema for set theory with ur-elements (including those involving arbitrary natural language vocabulary) will express truths — when we're talking about this favored set theory. But downgrade the necessity one takes the physical applications of this preferred set theory to have — or otherwise countenance a sense in which these set-theoretic constraints on how all predicates can apply are contingent. That is, countenance (physically and perhaps metaphysically impossible) scenarios in which the facts and laws about 'all possible ways of choosing' are different — so that different axioms for set theory with ur-elements reflect the 'laws of logical/combinatorial possibility' constraining how any predicates can apply to a given totality of objects. In the most extreme case this might involve countenancing — intuitively metaphysically impossible — scenarios in which there are 3 or 5 different ways for a predicate can apply (or fail to apply) to two objects. I will describe a less extreme case below.

But accepting above account of the set-theoretic analog to physical geometry creates immediate problems for Hamkins' platonistic multiverse. For it implies that any hierarchy of sets $V_@$ which is a candidate for reflecting the set-theoretic structure of the actual world must satisfy a non-modal version of the above comprehension principle.

$$\forall z \in V_@ \forall w_1 \forall w_2 \ldots \forall w_n \exists y \in V_@ \forall x[x \in y \Leftrightarrow ((x \in z) \wedge \phi)].$$

However, this implies that $V_@$ can't exist within any Hamkins' style Platonistic multiverse which contains an expanded hierarchy $V[G]$ for every hierarchy $V$ it contains.[16] However, much analogies with geometry inspire us to say that our notion of 'all possible ways of choosing' are somehow contingent, we still can't say *the actual world contains* both a hierarchy of sets, $V_@$ , which reflects the actual world's constraints on 'all possible ways of choosing', and a generic extension of that hierarchy $V_@[G]$.

[16]For, if $V[G]$ existed then $G$ would witness the violation of comprehension for $V_@$ since $G$ picks out a subset of $\mathbb{P}$ which $V$ doesn't already contain.

So, to summarize: a natural answer to 'what's the set-theoretic analog to questions about the geometry of the actual world/physical space we live in?' seems to fall out of treating set theory and geometry analogously but Hamkins can't give that answer.

## 5. The Best of Both Worlds?

With these problems for the idea that Hamkins' (Platonistic) multiverse treats set theory and geometry analogously in mind, let me conclude by saying a bit more about the approach to set theory (briefly sketched above), which I think *does* fall out of taking the analogy between set theory and geometry very seriously.

On this view, there are objective joint carving laws constraining how all predicates apply, much as (non-Humeans about laws would say) there are objective joint carving laws about what's physically or geometrically possible in the actual world. These laws are plausibly metaphysically necessary (though one might say they are merely physically necessary if one was willing to bite this bullet in order to secure maximum match with geometry).

These laws determine unique facts about what it takes for an iterative hierarchy to contain 'all possible subsets' at each layer. Hence there's a true/intended hierarchy of sets whose structure is determinate at least up to width; each layer of the true hierarchy contains all subsets of layers below.

However, there's a serious (genuinely set-theoretic, not merely formal or playful-reinterpretive) topic studied by people considering variant set theories, namely, what worlds in which the laws of logical possibility were different (so that there were more/fewer distinct ways a predicate could apply to some objects within a given plurality) would be like. Depending on whether we say the laws of logical possibility are metaphysically vs. merely physically necessary these possibilities may be:

- metaphysically possible ways reality could have been

- metaphysically impossible worlds of the kind already studied in hyperintensional metaphysics.

I prefer the latter take, and would like to note that there's already an independently motivated and developed literature on hyper-intensional modals [14, 4][17]. But, in either case, we wind up with an approach to set theory which interestingly combines features of realist and anti-realist approaches to set theory.

On one (realist) hand, there's an intended and physically important object of study corresponding to 'naive' iterative hierarchy set theory: the study of the laws of combinatorial possibility for the world we live in. There's a preferred hierarchy of sets (at least up to width) which reflects the logical/combinatorial constraints on all properties and objects in our reality. This true hierarchy (and the set theory describing it) reflects constraints on non-mathematical reality just as those of true geometry do: It would be physically (and perhaps metaphysically) impossible for a predicate to apply to some cats or sets of cats etc. without our hierarchy of sets already containing a corresponding set.

But, on the anti-realist hand, we have a robust subject matter to be studied by set theorists beyond obviously unintended (by Hamkins) countable models. This subject matter is almost exactly like the one Hamkins takes to be studied by geometry: what it would be like to (literally) live in a world where geometrical/set-theoretic structure of reality were different — and hence the facts about what's physically/geometrically possible for non-mathematical objects were different.

If we accept the proposal in §4.1 about what makes some set-theoretic universe the correct one for our reality/the actual world, then we get a tight connection between facts about this preferred hierarchy of sets and facts about how it's logically and (in some cases) physically possible for physical properties to apply to concrete objects like cats and spaceships. Just as we can make sense of Abbottonian [1] science

[17]Note that since we won't have FOL violations in any worlds in Hamkins multiverse, certain general concerns about impossible worlds and whether we'll have non-trivial counterpossible counterfavtuals may not apply

fiction novels about what it would be like to live in a world with a radically different geometry, we can make sense of Borghesian science fiction novels [15] about what it would be like to live in (physically or metaphysically impossible) worlds where the constraints on logical possibility are different.

For example, if CH is true at our world, studying forcing extensions could put us in touch with facts about what holds in metaphysically impossible worlds [14, 4] where the laws of logical possibility (i.e., the general combinatorial constraints on how any relations can apply to any objects) are different. These will be worlds in which (not only is the hierarchy of sets different) but it is actually in some sense possible for predicates to select certain subsets of the natural numbers[18] witnessing the falsehood of CH which aren't even possible in our universe.

This vindicates the idea that, just as there's the correct geometry for our reality, there's a correct set theory (at least up to width) for our reality.

5.1. **Another Problem.** Let me end by briefly mentioning a second problem for Hamkins' multiverse as currently stated which may be avoided by switching to the modal version of the multiverse suggested above. Hamkins' plenitudinous platonist formulation of the multiverse view makes it hard to cash out the contrast between fully grown hierarchies of sets (what we truly "interests us") and toy models.

For, Hamkins says that for every $V$ there's a $V'$ from whose perspective $V$ is countable. So, from the multiverse point of view, $V[G]$ is ultimately just as much a countable model as the traditional view's countable model $M$ in our background $V$. In this way the actual effect of adopting the existing plenitudnious Platonist form of the multiverse corresponds more to the formalist "tearing down" approach to the change in attitudes to geometry discussed above than the "building up" approach which we said fits with Hamkins descriptions of the change and the kind of motivations he cites for making it.

[18] Note that forcing extensions don't change facts about the natural numbers.

Hamkins' multiverse perspective seemingly corresponds to debunking the idea that any geometry could do what naive geometry was supposed to, and rescinding our sense that any the 'great circles on a sphere' interpretation of line was any worse (i.e., any more a "playful reinterpretation" rather than an acceptable interpretation) than any other interpretation of line.

In contrast, adopting the "best of both worlds" proposal above lets us give clear sense to this contrast. "Full grown hierarchies" of sets are ones that contain 'all possible subsets' in a way that makes their satisfaction of the full comprehension principle above a matter of metaphysicaly necessary logico-combinatorial law. The fully grown hierarchy of sets (pure or with ur-elements) for our universe (up to width) is the one that contains sets witnessing all logico-combinaorially possible ways for a predicate to apply to some objects within an infinite collection. just as the physically correct axioms of geometry are true when understood be quantifying over the physical points and lines that (in a law like manner) constrain all possible paths of objects through spacetime. And forcing tells us about alternative fully grown hierarchies of sets, in telling us about hierarchies that have this property (witnessing all logico-combinatorially possible way of choosing some objects from within a set that they contain), but exist in metaphysically impossible scenarios where the laws of first order logic remain truth preserving but the there are more or fewer possible ways that a predicate could apply to some objects within an infinite collection.

## 6. Conclusion

In this paper I've reviewed some key philosophically important features of Hamkins' multiverse program, argued the idea of treating set theory and geometry analogously, which Hamkins uses to motivate his Platonistic multiverse program actually raises a (prima facie) problem for that view. For taking this analogy seriously suggests that there should be some set-theoretic analog to questions about the geometry of physical space. And, I have argued, it also suggests a certain natural story about what that analog should be. However, well known controversial features of Hamkins' platonistic development of his multiverse program prevents us from accepting this story. And a replacement story seems hard to provide, for it's not clear that there's any other principled sense in which the structure of the actual world could pick out a preferred hierarchy of sets within Hamkins multiverse.

To fans of the analogy between set theory and geometry, my argument might seem to result in a stalemate. Hamkins' approach to set theory can't mirror there being a genuine (physical) fact of the matter about the parallel postulate. But standard realist approaches can't mirror the legitimacy of alternative geometries as objects of (serious) mathematical study. So no one can make analogs of both things we want to say about geometry come out true for set theory. However, I have suggested a compromise which might give the best of both worlds. On this view we identify a natural sense in which certain axioms of set theory reflect the true logical structure of the the actual world, much as certain axioms for geometries reflect its geometrical structure. But we honor Hamkins' arguments that studying set theories satisfying variant hierarchies of sets via forcing extensions has a legitimate 'genuinely set-theoretic' (as opposed to merely formalist or playful reinterpretation involving) subject matter. For we say that studying these different set theories illuminates counter-possible modal facts, about worlds in which set theory is different because certain general logical laws constraining *everything* are different[19]. We study what it would be like to live in a world where the laws of logical possibility were different, much as in studying different geometries we study what it would be like to live in a world where the laws of space were different.

The resulting view is formally and mathematically much like Hamkins' current proposal. However, some important philosophical differences should be noted. First, adopting the realist approach to an intended model of set theory gives one definite right answers to questions like CH and other proof transcendent facts about the

[19]A natural place to start with this program would be Hamkins and Linnebo's paper on modal logic [10].

sets (unlike on Hamkins' view). It also undermines advantages antirealist views like Hamkins' usually enjoy with regard to the Benacerraf problem [3, 7, 2]. Second, going all the way with the analogy between set theory and geometry as suggested above would mean saying that there's an important portion of pure mathematics which must be understood modally rather than as the study of some abstract objects.

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