THE ACCESS PROBLEM AND KNOWLEDGE OF LOGICAL POSSIBILITY

ABSTRACT. Accepting truth-value realism can seem to raise an explanatory problem: what can explain our accuracy about mathematics, i.e., the match between human psychology and objective mathematical facts? A range of current truth-value realist philosophies of mathematics allow one to reduce this access problem to a problem of explaining our accuracy about which mathematical practices are coherent – in a sense which can be cashed out in terms of logical possibility. However, our ability to recognize these facts about logical possibility poses its own access problem.

I propose a solution to this residual access problem. The key idea is that accepting powerful and correct general principles for reasoning about logical possibility can be the most efficient way to predict and explain the behavior of concrete objects. Although experience with the physical world is not needed to *justify* mathematical beliefs, I will suggest that our dealings with concrete objects can *explain* how we came to employ good a priori methods of reasoning about logical possibility.

1. INTRODUCTION

It's appealing to think that there are right answers to all arithmetical questions[1] and many questions in other mathematical domains – regardless of whether our proof practices will ever let us discover these answers. However, accepting such truth-value realism raises an explanatory problem. What can explain our true beliefs about mathematics, i.e., the match between human psychology and objective mathematical facts?

[1]That is, every sentence in the language of first order arithmetic, i.e., the language with function symbols $+$ and $\cdot$ and constant symbol 0, is either true or false.

The history of past attempts to respond to this problem can make it appear that no adequate explanation of human accuracy about mathematics is conceivable[2]. Accordingly, accepting truth-value realism about mathematics can seem to require positing an extra inexplicable coincidence. This objection is sometimes called the access problem[3].

A range of current truth-value realist philosophies of mathematics (views within I will call the 'structuralist consensus') allow one to reduce the this problem of explaining our accuracy about mathematics to a problem of explaining our accuracy about which mathematical practices are coherent[4] – in a sense which I will cash out using a logical possibility operator. However, this leaves us with a residual access problem concerning logical possibility.

I propose a solution to this residual access problem. The key idea is that accepting powerful and correct general principles for reasoning about logical possibility can be the most efficient way to predict and explain the behavior of concrete objects. Although experience with the physical world is not needed to *justify* mathematical beliefs, I will suggest that our dealings with concrete objects can *explain* how we came to employ good a priori methods of reasoning about logical possibility.

[2]For example, we can't appeal to causal contact with mathematical objects to explain the accuracy of our beliefs, because mathematical objects are causally inert. And we can't trivially explain accuracy about mathematics by saying literally any choice of mathematical axioms (including syntactically incoherent ones) will implicitly define mathematical vocabulary succeed in expressing truths, since this fits badly with actual mathematical practice and makes the scientific usefulness of mathematics a mystery.

[3]Classic formulations of the access problem, like Benacerraf's [2] have often targeted fairly traditional Platonist philosophies of mathematics. However, as we will see below, I think the best form of the access worry naturally extends to make trouble for other forms of truth-value realism as well.

[4]By this I mean semantically coherent, not just syntactically coherent.

One might wonder why approaching the access problem indirectly, through logical possibility, can help. Although I don't claim that no analogous argument could be given using other primitives, I think adopting the Structuralist Consensus and thus reformulating access worries in terms of logical possibility is helpful in two major ways.

First, we will see that the notion of logical possibility 'comes packaged' with particularly simple and direct expected connections to constraints on the behavior of concrete objects. In contrast, mathematical existence claims tend to be connected to facts about concrete objects via principles which are either more complicated (e.g. 'every true first order statement has a set model) or more conceptually negotiable (as witnessed by the famous incident where unexpected results in physics motivated a separation between geometry and physics rather than a revision to our theory of geometry).

Second, the apparent ontological commitments of mathematical existence claims can seem to raise an additional question about how we came to postulate the 'right' mathematical objects (as opposed to other coherent but 'wrong' objects). Approaching the access problem through logical possibility lets us bracket this question, while explaining how we came to have coherent mathematical practices[5].

2. Access Worries as Explanatory Demands

Now, let us consider what it means for a philosophy of mathematics to face an access problem, and what it would take to solve such a problem.

[5]This is not to say that one can't grasp the notion of logical possibility and then say wrong things about logical possibility (and hence mathematics). Indeed, the idea that you can is crucial to securing the truthvalue realist aims of the structuralist consensus (e.g., this is what lets us say that adding ¬con(PA) to our current arithmetical practice would be syntactically consistent but wrong). Allowing this is what generates the 'access problem for logical possibility' (how could we have come to have so many accurate beliefs about logical possibility?) which this paper aims to solve.

Benacerraf famously introduced an access worry for platonists expressed in terms of a (now widely rejected) causal theory of reference[2].

However most recent work, following Hartry Field[13], makes no appeal to such a constraint, and instead views access worries as arising from an unmet explanatory demand.

Various ways of cashing out this explanatory demand have been proposed. For example, Field himself says the realist must explain why, reliably, 'if mathematicians believe that P then P', and this seems hard to do in a way that is compatible with realism. Linnebo[30] says the realist must explain things like why, reliably, if mathematical sentences like "2+2=4" hadn't expressed truths, then people wouldn't have accepted them. I think we can do better, by construing access worries just in terms of an apparent commitment to *some* unexplained extra coincidence (beyond those required by otherwise attractive alternative approaches to the same domain)[6].

However, the subtle difference between these approaches will not matter for anything that follows. For, the explanatory strategy I propose will

[6]We have a general sense of what it would be for some regularity in the world to 'cry out for explanation.' And we tend to think that scientific or philosophical theories are, ceterus paribus, less attractive insofar as they posit regularities which cry out for explanation but include tenants which make satisfying explanation of these regularities impossible. I think that formulating access worries are most attractively understood as a specific instance of this general epistemic fact lets avoid the kind of trivialization worries discussed in [8]. (See REDACTED for more details)
Admittedly one might read Clarke-Doane's work in that paper as arguing (from philosophers' failure to 'cash out' the coincidence avoidance intuitions behind access worries in other, less controversial, terms) that such coincidence avoidance intuitions become unreliable when regularities involving necessary truths (like those of mathematics or logic are at issue). If true, this would be a problem for my formulation of the access problem as well. However, saying that coincidence avoidance intuitions are unreliable when applied to necessary truths (or that one can explain coincidences just by stapling together two unrelated modally robust explanations for either half of the coincidence) would require us to give up ubiquitous fruitful and well entrenched methodology within mathematics, which often uses intuitions about coincidence avoidance to guide research [1][28]. For example, the history of John Conway's 'Monstrous Moonshine' conjecture dramatically illustrates the important and fruitful role which looking for a deeper unifying explanation for striking match between mathematical facts *even though a proof of both facts already exists* plays in contemporary mathematical practice.Thus, I reject this criticism.

address both Field's and Linnebo's conditionals and banish any apparent commitment to an extra inexplicable coincidence.

Note that (on all these specifications), access worries turn out to be quite different from (and more troubling than) mere skepticism about mathematical objects and facts. For, access worries appear to reveal an internal tension in truth-value realists' overall theory of the world[7]. Also note that access worries are not (in the first instance) about justification (and may be resolved without providing any further evidence for the truth of our mathematical beliefs).

2.1. Access Worries As 'How Possibly' Questions, and How to Answer Them.

Regardless of which approach you favor, one can think about access worries as presenting a 'how possibly' question, "how can we have gotten significant reliability about realist mathematics?" Nozick and Cassim note that many philosophical questions take this form[5][34] and suggest one can think about how possibly questions as involving something thing like blocking conditions. In such cases, something seems impossible because of some (whether consciously recognized or not) obstacles[8]. And one can answer the how possibly question by giving an example explanation which is compatible with all the obstacles – even if there is little reason to believe that this explanation is true.

[7]Specifically, they seem to reveal a tension between the truthvalue realist's philosophy of mathematics and their more general beliefs such as which kinds of coincidences are implausible and what kind of explanations for regularities are plausible.

[8]Cassam writes, "...how-possible questions are obstacle-dependent questions. We ask how x is possible when there appears to be an obstacle to the existence of x. We don't ask how x is possible if there is no perceived obstacle or no inclination to suppose that x is possible. So, for example, we don't ask how baseball is possible or how round squares are possible. Where an obstacle-dependent how-possible question does arise there appear to be two basic strategies for dealing with it. .... A different approach would be to argue that freedom is possible even if all actions are causally determined, or that evil is possible even if God exists...What they deny is that the alleged obstacles are insuperable and, in this sense, genuine"[5]

Indeed, one can often best answer such how possibly questions by giving a *simplified* explanation which (thereby) is known to include some false elements – provided that all blocking conditions are accommodated and the core mechanisms involved are sufficiently plausible. For a (somewhat macabre) example, imagine someone who raises the following 'how possibly' question

> Historical records of the Irish potato consumption during the 1850s show that every time the price of potatoes increased, potato consumption *also increased*. How could this paradoxical seeming pattern have arisen?

One could attractively answer this 'how possibly question' (and dispel feelings of impossibility) by providing a simplified model like the following. Suppose every person in Ireland in 1850 has exactly $10 available each day to purchase food and there are only two kinds of food sold: potatoes and beef. Everyone needs 2200 calories a day, and prefers to eat beef to potatoes. Every 100 calories of beef cost $1 and, initially every 500 calories of potatoes cost $1. One can easily work out the math and see that each time the cost of potatoes rises the less money they have to buy beef so the more potatoes they buy despite the higher price.

Clearly this story contains many false elements; we know that actual Victorian shoppers behavior wasn't nearly so simple. But this unrealisticness actually helps the above story answer the 'how possibly' question (and dispel feelings of inexplicability), by more clearly presenting a core mechanism which accommodates all relevant blocking conditions and can plausibly be complicated and adapted to fit known historical details[9].

[9]See [22] for more on the value and nature of such unrealistic/idealizing explanations in the sciences.

I will propose an analogous simplified explanation for human accuracy about mathematics (as members of the Structuralist Consensus understand it). I take the blocking conditions here to include things like the abstractness and metaphysical necessity of mathematical facts, lack of causal contact with mathematical objects, and the existence of mathematical theories without scientific applications. These are the things which make an adequate explanation of mathematicians accuracy (conceived of either as Field or Linnebo conceives it) seem impossible. And I will attempt to dispel this impossibility intuition by giving a possible explanation which accommodates all these blocking conditions even if it simplifies in factually inaccurate ways.

Of course whether such a simplified explanation succeeds in answering a 'how possibly' question depends on the specific blocking conditions one takes to be part of that question. So someone could always re-raise the access problem for truth-value realism about mathematics by citing features of our actual phenomenology/biology/history which recreate the appearance that no explanation is possible. But I'm not aware of any way that my story differs from reality would seem to create such an impression.

3. Modal Structuralism and its Access Problem

3.1. **Structuralist Consensus.** A number of current philosophies of mathematics (forming, what I call the structuralist consensus) are committed to a close relationship between coherence and mathematical facts. Such views take mathematics to be 'the science of structure', and maintain that any choice of new mathematical structures coherently extending one's current mathematical practice would succeed[10]. I have in mind views like Mary Leng's fictionalism[29], Geoffery Hellman's ante rem structuralism[23], and

[10]Of course, it is the whole of one's practice that must be coherent. So even coherent posits might be unacceptable if they are not jointly coherent with existing posits.

Quantifier Variance fueled neo-Carnapian realism about mathematical objects[11][12].

As these views allow any coherent mathematical structure to be posited, they transform an explanation of our access to coherence facts into an explanation of our access to mathematical knowledge. For if we take our ability to postulate coherent, rather than incoherent, mathematical structures for granted, mathematical knowledge flows simply by making various logical inferences.

Thus adopting one of these views promises to let us address access worries, if only we can solve the residual access worry about how we manage to recognize coherent mathematical posits (and coherent extensions of our current mathematical practice).

For concreteness, in the bulk of this paper I will focus on how this residual access problem arises for Modal Structuralism and whether it can be dispelled. However, I take the solution proposed to generalize, since essentially the same kind of knowledge of coherence needs to be explained by all members of the structuralist consensus.

3.2. **Logical Possibility.** We seem to have an intuitive notion of logical possibility which applies to claims like $(\exists x)(red(x) \wedge round(x))$ and makes sentences like the following come out true.

- It is logically possible that $(\exists x)(red(x) \wedge round(x))$
- It is not logically possible that $(\exists x)(red(x) \wedge \neg red(x))$
- It is logically necessary that $(\forall x)(red(x)) \to \neg(\exists x)(\neg red(x))$.

[11] I have in mind views like [24] and [42][43] and my own slightly more meta-ontological realism friendly proposal in REDACTED.

[12] Some of these views of the semantics of mathematical claims are hermeneutic and others are revisionary, to use Burgess and Rosen's terminology [3]. (Hermeneutic views present accounts of what we actually mean, revisionary views present accounts of what we should mean/how we should revise our practices.) I won't stress the difference here, because it won't mater to the genesis or solution of the access worries discussed here.

Philosophers representing a range of different views of mathematics have made use of this notion[13] and are comfortable applying it to non-first order sentences[14]. Like Hartry Field, I take take it to be a primitive concept[15] not reducible to any facts about set theoretic models or possible worlds.

Modal Structuralists, like Hellman, take the true content of a pure mathematical claim to be a claim about logical possibility like $(\Diamond D) \wedge \Box(D \rightarrow \phi)$[16]. For example, let PA_2 be the sentence in second order logic which says that the objects satisfying some relation $\mathbb{N}$ satisfy the second order peano axioms, taking S to be the successor relation (note that all quantifiers are restricted to objects satisfying $\mathbb{N}$). And let ϕ is a sentence in the language of arithmetic (so we can also suppose that all of its quantifiers are restricted to the objects satisfying $\mathbb{N}$). Then the modal structuralist can render the intended meaning of mathematicians apparent assertion of ϕ as $\Diamond(PA_2) \wedge \Box(PA_2 \rightarrow \phi)$[17].

If one accepts Modal Structuralism, then our knowledge of mathematics can be explained by appeal to our knowledge of logical possibility. But there's still an obvious and deeply analogous remaining worry concerning our knowledge of the (often complex and powerful) logical possibility claims need to make sense of mathematics.

[13]See the discussion of the corresponding notion of consequence in [13],[38] and [23].

[14]If you are skeptical that there is such a notion, note that it is definable in terms of the even more common notion of validity (something is logically possible iff its negation is not logically necessary iff the inference from the empty premises to its negation is not valid).

[15]At first glance, one might be tempted to identify claims about logical possibility with claims about the existence of a set theoretic model. But see [15] 2.3 for what I think is a convincing argument against this view.

[16]Hellman treats set theory slightly differently, because of special issues about the height of the hierarchy of sets. See [23] for details on this and REDACTED for details on how to extend the streamlining of Hellman's core proposal discussed here to his preferred story about set theory.

[17]Note that reference to properties and relations like $mathbb(N)$ and S can be replaced by other non-mathematical relations with the same arity e.g. 'cat()' and 'admires(), following Putnam's suggestion in [36]. Hellman handles this point slightly differently in [23], but the difference makes no difference here.

As facts about logical possibility are no more directly observable than facts about mathematical objects, adopting modal structuralism simply replaces one daunting access problem with another[18]. To appreciate the nature and difficulty of solving this access problem, note that recognizing facts about logical possibility requires something more than using the introduction and elimination rules for the first order logical vocabulary[19].

In what follows I will offer a *simplified story* (which avoids blocking conditions) about how creatures broadly like us could have gotten mathematical knowledge. This story will proceed by offering an account of how we could have come to have substantially true beliefs about logical possibility and turned this into mathematical knowledge by positing *coherent* mathematical structures.

3.3. **Streamlining Modal Structuralism.** But first I must show that it is possible to streamline existing formulations of Modal Structuralism in a certain helpful way. These formulations use second order logical sentences like PA_2, the second order Peano axioms, above (under the $\Diamond$ of logical possibility) to pin down the intended behavior of relevant mathematical structures. But if continue to do this, we will need a separate explanation of our access to facts about second order quantifiers[20]. Also, if one takes second order existence claims outside the $\Diamond$ to generate ontological commitment, then

[18] See [39] for development of the slightly different problem of accounting for accuracy about logic, e.g., how we could have come to use correct expressions like the first order logical connectives $\wedge, \vee, \neg, \exists, \forall$ with their usual introduction and elimination rule, rather than how we could come to know logical possibility facts involving these objects.

[19] Admittedly, by Gödel's completeness theorem, a first order logical statement requires something logically possible iff it is syntactically consistent [20] (though this fact itself can't be reliably captured in a first order fashion). However, this no longer holds if ϕ itself contains applications of $\Diamond/\Box$ or some other non-first order vocabulary (as is needed to categorically describe mathematical structures like the natural numbers).

[20] We would need a story about how we can be accurate about whether a concrete scenario satisfies a second order description.

employing second order logic threatens to re-raise the issues about object existence which focusing on logical possibility promised to let us bracket[21].

I'll avoid these problems by introducing a notion of logical possibility *holding certain facts fixed* which will let us express Hellman's nominalistic mathematical sentences using only first order logic and the conditional logical possibility operator. Making this change also brings out an intrinsic unity in the facts we want to explain human accuracy about. This saves us the trouble of separately explaining access to facts about logical possibility and second order logic.

To introduce this generalized notion of conditional logical possibility consider a sentence like, "Given what cats and baskets there are, it is logically impossible that each cat slept in a distinct basket." There's an intuitive reading on which this sentence will be true if and only if there are more cats than baskets[22]. This reading employs a notion of logical possibility *holding certain facts fixed* (in this case, structural facts about what cats and baskets there are[23]).

I will use a conditional logical possibility operator $\Diamond$, which takes a sentence ϕ and a finite[24] (potentially empty) list of relations $R_1...R_n$ and produces a sentence $\Diamond(R_1...R_n)\phi$ which says that it is logically possible for ϕ to be true, given how the relations $R_1...R_n$ apply. For ease of reading, I will sink the specification of relevant relations into the subscript as follows: $\Diamond_{R_1...R_n}\phi$.

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[21]Second order quantification is usually taken to require accepting a comprehension principle which applies to actual objects as under the $\Diamond$ of logical possibility.

[22]Admittedly, there's another reading of this sentence on which it expresses a necessary falsehood. However, this is not the reading I have in mind.

[23]Hellman's own use of logical possibility given the material facts commits him to the coherence of something very much like this notion.

[24]I don't suggest that all meaningful claims involving the concept of logical possibility can be expressed in this form. For example, you might well think propositions subscripting infinite collections also makes sense.

Thus, for example, the claim, 'Given what cats and baskets there are, it is logically impossible that each cat slept in a distinct basket' becomes:

(CATS)

$$\Box_{\text{cat,basket}}\neg\Big((\forall x)\Big[\text{cat}(x) \to (\exists y)\,(\text{basket}(y) \wedge \text{sleptIn}(x,y))\Big] \wedge (\forall z)(\forall w)(\forall w')\Big[\text{basket}(z) \wedge \text{cat}(w) \wedge \text{cat}(w') \wedge \text{sleptIn}(w,z) \wedge \text{sleptIn}(w',z) \to w = w'\Big]\Big)$$

Remember that when evaluating logical possibility we consider all possibilities for the relations mentioned in the statement under consideration, whether we can describe them or not. This is analogous to requiring second order quantifiers to range over all possible collections. And this is the key fact which will let us rewrite second order logical descriptions of mathematical structures as with claims about how it would be logically possible for some arbitrarily chosen relations to apply[25].

[25]Also note that one can perhaps get correct truth conditions by thinking of $\Diamond_{R_1\ldots R_n}\phi$ claims as holding fixed the *particular objects* in the extension of the relations $R_1 \ldots R_n$ – and then asking *de re*, of these objects, whether one could supplement them with other objects (and choose extensions for all other relations) so as to make ϕ true. However, there seems to be a primitive notion of preserving the *structural facts* about how some relations apply, which does not depend on our understanding any such controversial *de re* claims. For example, in the case of CATS above these will be scenarios which agree with the actual world on: the number of objects satisfying cat(), the number satisfying basket() and the number of things in the extension of both cat() and basket(). However, preserving the structural facts does not require preserving facts about identity (or supposing that the relevant 'cross logical-possibility counterparthood' facts are well defined).
If, say, one cat died and an additional kitten was born, these structural facts would remain unaltered. Crudely, we might gesture at the idea of preserving the structural facts by saying that two scenarios have the same structural facts about the relations $R_1, \ldots, R_n$ if the objects satisfying some R_i in the first scenario (more precisely those x such that $\exists y_1, \ldots, y_k, y_{k+2}, \ldots, y_m R_i(y_1, \ldots, y_k, x, y_{k+2}, \ldots, y_m)$ for some i, k and m.) are 'isomorphic' (under $R_1, \ldots, R_n$) to the objects satisfying some R_i in the second scenario. Note that this is not intended to be a definition of the concept, only an attempt to point at the correct primitive notion, as the very notion of isomorphism would be defined in terms of logically possible mappings.

For example, using conditional logical possibility, we can express claims like the induction axiom for number theory (which is usually expressed in second order logic) as follows.

> **Induction Axiom**: if some property applies to 0 [26] and to the successor of every number it applies to, then it applies to all the numbers.

- **Induct**: 'It is logically necessary, given how number and successor apply, that if 0 is happy and the successor of every happy number is happy then every number is happy.' i.e., $\Box_{\mathbb{N},S}$(if 0 is happy and the successor of every happy number is happy then every number is happy).

Thus, we can write a sentence $PA_\Diamond$, which categorically describes the natural numbers [27] and hence ensures that for every sentence of number theory ϕ, either ϕ or $\neg\phi$ is a logically necessary consequence of $PA_\Diamond$, i.e., $\Box(PA_\Diamond \rightarrow \phi)$ or $\Box(PA_\Diamond \rightarrow \neg\phi)$. Note that while PA_2 is traditionally expressed in terms of the relations number, successor etc. we can substitute any other relations of the same arity (happy, loves etc.). As logical possibility ignores any particular features of relations (unless conditioned on) the truth value will be unaffected[28].

We can also make nested claims about logical possibility. Note that in a nested claim with the form $\Diamond\Box_R\psi$, the subscript freezes the facts about how the relation R applies in the scenario being considered, which may *not*

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[26]By '0' I mean the unique number which is not the successor of any number, and I take the rest of the sentence to be spelled out using this definite description in standard Russellian fashion.

[27]Essentially, $PA_\Diamond$ conjoins the finitely many axioms of first order PA-Induction with the statement of induction in terms of logical possibility (Induct) above. The only other modification needed is to replace mathematical vocabulary with non-mathematical vocabulary ('number', 'successor' etc.) with non-mathematical vocabulary of the same arity, as per Putnam's suggestion above.

[28]So, for example, the sentence $\Diamond(\exists x)(\exists y)(\text{Dog}(x) \wedge \text{Cat}(y) \wedge \neg x = y)$ and the sentence $\Diamond(\exists x)(\exists y)(\text{Dog}(x) \wedge \text{Lemur}(y) \wedge \neg x = y)$ always have the same truth value.

be the state of affairs in the actual world. So, for example, POSSIBLY CATS (below) expresses a metaphysically necessary truth. For, whatever the actual world is like, it will always be logically possible for there to be, say, 3 cats and 2 baskets, and this scenario is one in which it is logically necessary (holding fixed what cats and baskets there are) that: if each cat slept in a basket then multiple cats slept in the same basket. So it is metaphysically necessary that POSSIBLY CATS.

(POSSIBLY CATS)

$$\Diamond\Box_{\text{cat,basket}}\neg\Big((\forall x)\Big[\text{cat}(x) \to (\exists y)\,(\text{basket}(y) \wedge \text{sleptIn}(x,y))\,\Big]\wedge$$
$$(\forall z)(\forall w)(\forall w')\Big[\text{basket}(z) \wedge \text{cat}(w) \wedge \text{cat}(w')\wedge$$
$$\text{sleptIn}(w,z) \wedge \text{sleptIn}(w',z) \to w = w'\Big]\Big)$$

While I take logical possibility to be a primitive (as Hellman does), in appendix A I explain how familiar set theoretic vocabulary can approximately mimic truth conditions for conditional logical possibility.

Using these nested claims about logical possibility (i.e., asserting the logical possibility of scenarios which are themselves described in terms of logical possibility) and the sentence $PA_\Diamond$ generated above, we can write a version of Hellman's paraphrase, $\Diamond(PA_2) \wedge \Box(PA_2 \to \phi)$, which avoids second order quantification: $\Diamond(PA_\Diamond) \wedge \Box(PA_\Diamond \to \phi)$. To see how notion of conditional logical possibility can replace second order quantification more generally, see appendix B.

4. Knowledge of Logical Possibility

Let us now turn to the challenge of explaining human accuracy about the logical possibility facts mentioned above.

In this section, I will propose a story about how creatures initially equipped with only the kind of non-mathematical faculties philosophers pressing an access worry are willing to grant (e.g., observation, first order logical deduction, scientific induction) might have developed good methods of reasoning about logical possibility. To appreciate the scope of what needs to be explained, note that merely using the correct introduction and elimination rules for first order logic does not allow one to recognize the positive fact that a scenario is logically coherent. For example, first order introduction and elimination rules don't allow one to recognize that it would be logically coherent for there to be two distinct things $(\exists x)(\exists y)\neg x = y$[29].

As inquirers, we attempt to predict and explain the behavior of concrete objects. There are more and less economical ways of doing so. When we are dealing with sufficiently diverse and plentiful collections of concrete objects, the most economical explanations for regularities may well appeal to a combination of general principles which constrain how any objects can be related by any relations, and specific physical or metaphysical laws whose application is restricted to certain particular kinds of objects or relations. I will suggest that, in this way, pressure to efficiently predict physical events in situations of evolutionary interest can help explain how creatures like us could have gotten a concept of logical possibility and then developed powerful methods of reasoning with it.

My story begins with the idea that our compositional language admits many different-looking representations whose falsehood is guaranteed by

[29]By Gödel's completeness theorem[20] it turn out that every logically incoherent first order scenario allows for a derivation of contradiction using the usual inference rules for first order logic. But this was a substantive result which it took real mathematics to prove, so not something the denizens of our story would or could assume. Furthermore, humans clearly don't infer that a scenario is logically coherent by checking all possible proofs (whose premises are true in the scenario) for a contradiction.

logic alone[30]. Thus many plans which we can verbally represent can be discarded as unrealizable purely on the grounds that they require something logically impossible. And there are practical benefits to be gained from being able to systematically recognize and focus our attention on those plans which are, at least, logically possible[31].

Accordingly I think it would be unsurprising if we eventually (either consciously, unconsciously or at the level of evolutionary selection) began to exploit the fact that certain linguistic patterns yield falsehoods no matter the content of the relations being represented – and acquired (something like) a notion of logical possibility including the following two principles (and the expectation that $\Diamond$ facts should follow elegant general laws).

- $\phi \rightarrow \Diamond_{R_1 \ldots R_n} \phi$
- $\Diamond \phi_{R_1 \ldots R_n} \leftrightarrow \Diamond \phi_{R_1 \ldots R_n}[S_1/S'_1 \ldots S_m/S'_m]$ where none of the S_i, S'_i are among the R_i

The first principle embodies the idea that we are talking about a notion of possibility, saying that everything actual is logically possible possible. The second idea embodies the idea that we are talking about possibility with respect to logical form alone, so that systematically replacing one relation

[30]One might naturally wonder why scenarios which we can predict won't be realized are ever entertained at all. Wouldn't it be better if we couldn't even formulate the idea of such scenarios? Surely it would be, if we could filter out impossible scenarios with little cost. But this is plausibly forbidden by computational complexity considerations. For obvious reasons it is useful to consider conjunctions, disjunctions and negations of scenarios which means that any such filtering would be, at a minimum, required to do the equivalent of filer out all unsatisfiable boolean formulas. Yet, if $P \neq NP$ then by the Cook-Levin theorem [10] then no polynomial time algorithm can accomplish such filtering. These considerations render it very plausible that we would be inclined to consider a great many scenarios (such as unsatisfiable boolean combinations) many of which we may later be able identify as belonging to particular class of scenarios which will never be realized. Thanks to REDACTED for making this cute point.

[31]Note that even though creatures with first order logic will already be disposed to reject plans when they derive a contradiction from them, there are further benefits to be gained from having a positive theory (e.g., being able to infer that one scenario is logically possible only if another one is, allows one to skip searching for a contradiction in the former scenario after seeing the later scenario realized).

with another doesn't change logical possibility facts. Note that many natural variants on this initial conception of logical possibility would intuitively count as getting something else right (e.g., setting out to learn facts about physical, chemical, metaphysical, or psychological possibility) rather than getting logical possibility wrong.

I will now attempt to explain how creatures (with familiar human observational, scientific-inductive etc. faculties but no mathematical knowledge) could go from having this kind of minimal conception of logical possibility to having powerful methods of reasoning about logical possibility sufficient to capture all of contemporary mathematics (understood in a modal structuralist vein).

4.1. **Three Mechanisms of Correction.** I will propose three key ways in which dealings with concrete objects can (directly or indirectly) help correct and accurately extend our methods reasoning about logical possibility.

4.1.1. *From ϕ to $\Diamond\phi$ and $\Diamond_{R_1...R_n}\phi$ facts.* Recognizing relationships between concrete objects can push us to accept some $\Diamond\phi$ statements. Imagine that you aren't sure whether the state of affairs described by some mathematical hypothesis involving relations P, Q, and R is logically possible. If I then point out that the relations of friendship, nephew-hood and having been in military service together apply in just this way to the royal family of Sweden, this will cause you to accept that the scenario in question is logically possible.

Similarly, recognizing actual relationships between concrete objects can create systematic pressure to accept particular claims about subscripted logical possibility. Just as what is actual is logically possible, what is actual is logically possible given any facts about the actual world [32]. Thus, for example, one can go from 'every dog loves some human' to '$\Diamond_{dog}$ every dog loves

[32]That is, for any collection of relations $R_1...R_n$ and state of affairs ϕ, if ϕ then it is logically possible that ϕ given the facts about how relations $R_1...R_n$.

some human'[33] or '$\Diamond_{human}$ every dog loves some human' or '$\Diamond_{dog,human,loves}$ every dog loves some human'. In this way recognition of actual relationships can also create pressure to accept certain $\Diamond_{R_1 \ldots R_n}\phi$ claims.

The advantages to be gained by recognizing useful physically possible scenarios can also create pressure to accept *general inference methods*[34] which allow one to recognize hitherto unrealized scenarios as logical possibilities. As a result, the benefits to be gained from recognizing physical possibility facts can push us towards methods of reasoning which allow us to arrive at the logical possibility of non-actual states of affairs. Similarly, it can also be useful to recognize what is possible while keeping certain relations fixed, and this can help explain our tendency to accept general inference methods which let one reliably derive true claims about conditional logical possibility.

For instance, consider someone who didn't accept (even finitary) choice as a valid inference method for logical possibility. That is, they weren't willing to infer from $(\forall x)(D(x) \rightarrow (\exists y)R(x,y))$ to $\Diamond_{R,D}(\forall x)(\forall y)(F(x,y) \rightarrow R(x,y)) \wedge (\forall x)(D(x) \rightarrow (\exists! y)F(x,y))$. Such an individual might know that the enemy has divided their army up into platoons (so $D(x)$ is true just if x is a platoon in their army) and know that every platoon had at least one soldier ($R(x,y)$ holds just if y is a soldier in platoon x) but yet be unsure if it was (even logically) possible for the enemy to select a single soldier in each platoon to be the platoon leader. Failing to recognize such a possibility would be disadvantageous, and the fact that in every circumstance where the question arose there was always such a choice relation would create pressure to accept such an inference procedure. Admittedly, this is a somewhat simple and contrived example but (as with mathematics) reasoning about

[33]Remember that this means considering a scenario which preserves the number of dogs in the actual world.

[34]I take these to include things like the use of inference schemas and various ways of manipulating mental and physical pictures.

logical possibility really comes into its own when we put multiple inferences together to reach more complex conclusions.

4.1.2. *From $\neg\phi$ facts to $\neg\Diamond\phi$ and $\neg\Diamond_{R_1\ldots R_n}\phi$ claims.* Even though the non-actual need not be non-possible, our need to elegantly explain regularities in the concrete world creates pressure to conclude certain states of affairs are logically impossible. Suppose, for example, that someone thought it was logically possible for 9 items to differ from one another in which of three properties they had, e.g., for 9 people to choose different combinations of sundae toppings from a sundae bar containing three toppings. This person would have to explain the striking law-like regularity that, regardless of the type of items and properties in question, we never wind up observing more than 8 such items. They might postulate new physical regularities to explain why apparently random processes of flipping three coins never generated the forbidden 9th possible outcome. However, this explanation (or some analogous one) would have to apply at every physical scale we can observe, from relationships between the tiniest particles to relationships between planets and stars (as well as to less concrete objects like poems and countries). A much more elegant explanation is that the unrealized outcome is logically impossible. Recognizing that the forbidden 9th outcome is forbidden in all possible domains is much more efficient than hypothesizing separate laws prohibiting it in each specific situation (and thus there is pressure to do so).

This mechanism also provides pressure to accept conditional logical possibility claims. For example, if we keep noticing that when there are 4 cats and 3 baskets it is never the case that each cat slept on a different basket, the most elegant explanation for this is that it would be logically impossible for each cat to have slept on a different basket.

Accordingly, we can think of facts about what's actual as simultaneously a useful source of data about what's logically possible, physically possible, chemically possible, etc. We try to efficiently predict what will happen by patching together laws with different levels of generality. Though (in principle) we always face a choice about whether to explain a given regularity in terms of logical necessity, physical law, metaphysical necessity or mere ceterus paribus regularity, patterns in our experience can still motivate attributing a noted regularity to logical necessity rather than physical law. For, as noted above, if a regularity holds as a matter of logical necessity, we should expect to see that all substitution instances of it (i.e., all sentences with the same logical structure) are true, whereas we would expect the opposite if some principle holds as a matter of merely metaphysical necessity or physical necessity. This is not to say that we always make the right judgment, but in the long term we face significant pressure to correct our mistakes.

4.1.3. *Approaching Reflective Equilibrium.* Finally, one should note that the pressures mentioned above don't exist in isolation. Rather the resulting beliefs (and inference methods) will be further corrected by interaction with one another. If one accepts the above story about how we could have gotten some initial 'data points' about logical possibility from our knowledge of the concrete world, one can then appeal to familiar processes of reflecting on our beliefs and recognizing when they conflict or cohere with one another to explain some further improvements in our accuracy[35][36].

Once some methods of reasoning come to strike us as initially attractive via the two mechanisms above, we can arrive at new more powerful laws (just as we do in the sciences) by considering how they unify and explain these methods of reasoning. For example, in the literature on the search for new axioms in set theory it has often been argued that we can reliably add new axioms by choosing principles which unify and explain the mathematical beliefs which we already have[27]. As Gödel puts it, "There might exist axioms so abundant in their verifiable consequences, shedding so much light upon a whole field, and yielding such powerful methods for solving problems... that, no matter whether or not they are intrinsically necessary, they would have to be accepted at least in the same sense as any well-established physical theory"[21]. If this is true, then it also seems plausible that the creatures in our just so story might reliably expand an initial collection of good methods of reasoning about logical possibility in the same way. Moreover, when we make incorrect generalizations these can be corrected by coming into conflict with well-entrenched and concretely motivated general principles.

Finally, remember that that the kind of elegant generalization which we see in the sciences (and which I want to invoke) goes beyond simple inferences

[35]While the notion of reflective equilibrium is frequently invoked to provide justification for a theory, I also take it to be widely presumed that it's a reliable process when given some initial degree of accuracy. Even philosophers who critique the use of reflective equilibrium to address access worries (such as Justin Clarke-Doane[7]) tend to formulate their objection as kind of 'garbage in garbage out' worry. Such philosophers seem to accept the reliability of induction and reflective equilibrium *if some initial accuracy about mathematics/morals could be generated*, but argue that it's puzzling how we could get initial basis of accuracy about particular mathematical and moral facts for reflective equilibrium to operate on.

[36]This process of reflective equilibrium relies on our ability to make valid logical inferences (though not our ability to recognize logical possibility). However, as noted above, we can take initial proficiency with first order (classical) logical vocabulary for granted, since solving the access problem for truth-value realist philosophies of mathematics merely requires explaining the reliability of our mathematical beliefs *given* widely accepted background assumptions shared by truth-value realists and anti-realists alike.

like 'the sun rose every day for the past billion years, so it will rise tomorrow.' It can include the kind of, seemingly astonishing, leaps we see in the sciences like going from observations of points of light in the night sky to a whole model of how the planets are arranged.

4.2. **Room for A Priority and Innateness.** One might worry that the mechanisms of correction my story considers could only explain a posteriori knowledge of mathematics, while our mathematical knowledge is generally assumed to be a priori. In response to this worry, I'd like to note that correction by experience playing an important causal role in explaining our mathematical knowledge does not prevent that knowledge from seeming to us to be (or even being) a priori. Interestingly, there seem to be plenty of examples in actual history where (in what might be playfully called 'epistemic Stockholm syndrome') conscious experience forces us to believe something, and then we decide that we should have reasoned that way all along. See [44] for an intriguing real life example of this phenomenon: The New York Times employed a computer simulation using a random number generator to change readers opinions about which a priori methods of reasoning about the Monty Haul problem (i.e., experience can change what methods of reasoning about probabilities people find immediately compelling *and* what methods think they should have accepted a priori). Or remember Mill's idea that [33] children learn about arithmetic by dealing with concreta and then expect these laws to hold with metaphysical necessity.

A related concern may arise for those who take our mathematical knowledge to face significant innate constraints[37]. Someone with this perspective

[37]See Spelke's experiments with infants, [41] for an example of the kind of data which might suggest that some reasoning about what patterns of relationships between objects are (something like) logically possible are relatively innate. Further results along these lines might suggest that children have good methods of reasoning about logical possibility before they are in a position to do much personal experimentation with concrete objects, or hear good methods of reasoning advocated in the classroom. Also see [11] and [4]

might find any just-so story which didn't explain how we came to have innate dispositions to reason correctly about mathematics unconvincing. In response to this concern, I'd like to note that (something like) the first two mechanisms of correction above could have been realized at an evolutionary level, rendering our dispositions to accept good mathematical reasoning innate. Though evolution may not care about elegance and theoretical economy in quite the sense that we do, mental resources are expensive and those methods of reasoning that could be encoded in the simplest manner and handle the most general situations would be favored[38].

5. Underdetermination by Evidence Worries

I will now consider a family of objections to the story I have told above. These objections arise from the following simple idea: we causally interact with relatively small collections of objects, but one needs to appeal to accuracy about the logical possibility of much larger (typically infinite) collections to explain our accuracy about mathematics. In this section I will address a number of under-determination worries which arise from this apparent gap.

5.1. **Scientific Induction Unreliable in Mathematics?** First, one might worry that scientific-induction style generalization from cases (whether it be

for more general empirical work relevant to the possible innateness of our logical and/or mathematical concepts.

[38]Note that although my simple model had generalization and correction happening on a single level (conscious, evolutionary, historical) one can imagine a two layer picture, where evolution nudges our minds to favor certain kinds of reasoning, and then more conscious learning from experience and rational equilibrium corrects our intuitions, training us out of following some (e.g., a feeling that it would be pretty weird if the Hilbert's hotel situation was logically possible) and strengthening others (much as it trains us out of following logical fallacies). Thus I think my model can easily accommodate the special felt obviousness of certain mathematical judgments which Parsons emphasizes in [35] together with the fact (if it is a fact) that no possible course of experience would lead us to give up certain of these logico-mathematical judgments.

implemented consciously, unconsciously or evolutionary) is completely unreliable with regard to mathematics. If this were correct, it would certainly raise a problem for my proposal that dealings with small concrete collections could have pushed us to develop accurate general methods of reasoning about logical possibility.

However, there's strong independent reason to reject insinuations that generalization from cases is completely unreliable in mathematics. Mathematicians frequently use hunches developed from past experience, judgments of general plausibility or theoretical attractiveness and the results of computational searches[39] to guide their research. For example, belief that Fermat's last theorem was true *before* a proof found was motivated by consistent failure to find a counterexample. If we want to make sense of the apparent success of this aspect of mathematical practice, we can't suppose that something about the nature of mathematics makes the kind of elegant generalization from cases we find in the sciences *completely unreliable* when applied to the mathematical realm[40].

5.2. **A Gap Between the Finite and the Infinite?** Next, one might worry that the story suggested above cannot explain the *degree* of mathematical knowledge we take ourselves to have.

[39]Of course, they do not do this naively. If they already know that counterexamples would have to be huge they wouldn't change their judgments because no small counterexamples were found.

[40]Of course, this is not to say that mistakes in generalizing from the patterns of how relations can apply to (often concrete) non-mathematical objects to accurate general principles about logical possibility is infallible. For example, one might argue that Freiling's argument [17] against the Continuum Hypothesis illegitimately transfers intuitions about physical space to constraints 0on logical possibility and set theory.

In this subsection I'll discuss the specific worry that my mechanisms of correction could explain human accuracy about logical possibility facts involving finite collections, but not accuracy about the logical possibility of even the smallest infinite collections, like the natural numbers[41].

To address this worry, I will make two points. First, the physical world seems to be (at least) helpfully describable in terms of some (countably) infinite collections like the above. Consider, for example, the stretches of space along the path of an arrow, or the stretches of time during which the arrow is traveling. Plausibly this can create some pressure to acknowledge the logical possibility of certain kinds of (small) infinite systems, and to avoid unreliable reasoning about what these systems must be like[42].

I admit that the existence of such stretches of space is controversial (especially with philosophers of an empiricist or materialist persuasion) and might even be considered an open scientific question. But I want to suggest whether or not such 'quasi-physical' objects as spatial paths actually exist[43], it is clearly very useful for us to think about space in terms of them in many contexts. And it is plausible that the mere practical usefulness of thinking in terms of certain structures is a somewhat reliable (if not infallible, like the actual-to-possible inference) sign of logical possibility. And this is all we need to explain how the usefulness of talking in terms of some non-concrete

[41]See Frege's [16] pg. 16 for a version of this objection. He suggests that different numbers are like different geological strata and that one cannot infer facts about one from the other.

[42]As Penelope Maddy emphasizes in [32] science and philosophy of science may leave us with real uncertainty about what logico-mathematical structure to ascribe to a physical system and nothing I say here is incompatible with this observation.

[43]Note that my project of using experiences with non-mathematical objects (along with knowledge of first order logic, ability to do scientific induction etc.) to *solve the access problem* differs crucially from historical attempts to reconcile mathematical truth-value realism with some empiricist thesis requiring all non-logical knowledge to come from the senses (or some materialist thesis saying that all objects must be material, which would rule out the existence of such chunks of space).

structures can be helpful in explaining our accuracy about logical possibility would be non-coincidental[44]

Second, even if you don't accept that we have access to any infinite physical collections, reasoning about how it would be logically possible for physical objects to be supplemented with an infinite collection of abstract objects can be very useful in stating elegant laws which predict and explain the behavior of physical objects.

Consider the task of predicting what physical inscriptions of series of letters one will ever encounter. In making these predictions, it can be helpful to imagine actual physical inscriptions existing alongside a larger system of abstract objects ('strings') which witness all logically possible ways putting together finitely many letter inscriptions chosen from the relevant finite alphabet.

Even if all the inscriptions we encounter are relatively short, the most efficient way for us to recognize patterns in what inscriptions are physically possible can plausibly involve recognizing the logical possibility of strings of arbitrary finite size. This is because many 'closure principles' which smoothly (help) predict the facts about what short strings are physically possible will have the consequence that very long strings are logically possible - even strings which are too long to physically realize given the number of fundamental particles in the universe.

Take, for example, the principle that for any logically possible inscription it is logically possible for there to be a 'doubled' inscription which concatenates that inscription with itself. Given the truth of principles like this, a scenario in which there are objects witnessing all logically possible choices

[44]In a sense this aspect of my story is like Quine's proposal that scientific usefulness of mathematical structures is a guide to mathematical actuality, but I am instead making the much less controversial idea that the usefulness of abstract structures is a good guide to their logical possibility.

of how to concatenate letters will be a scenario in which there are infinitely many different abstract objects. Our methods of reasoning about logical possibility for infinite collections can be tested and corrected by the consequences they have for what this infinite collection of all strings would have to be like (and thereby, indirectly, for what physical string inscriptions are possible)[45].

Accordingly, I think there is an adequate explanation of how dealing with finite numbers of physical objects could have lead us to recognize facts about the logical possibility of infinite collections.

5.3. **Logical Possibility and Large Collections.** A final worry concerns our access to facts about logical possibility involving larger infinite collections. Perhaps one can explain our accuracy in reasoning about countably infinite collections as above. Yet capturing intuitively correct truth conditions for statements of set theory (via the structuralist consensus) requires evaluating claims about the logical possibility of scenarios involving uncountably many objects[46]. Thus, one might worry that principles of reasoning which are shaped to elegantly predict and explain what is logically possible for finite and countably infinite collections cannot account for the degree of logical (and hence mathematical) knowledge which we actually have.

[45] Admittedly the presumption that logico-mathematical reality is (in some sense) uniform plays a key role here. I think the very same expectation that reality is simple and uniform and hesitance to posit coincidences which drives access worries, tells us to expect that logical possibility facts are uniform. We can, of course, learn that large collections fail to be like small ones in certain specific ways (c.f. Hilbert's hotel) by using our methods which effectively do presume that large collections are like small ones. But I think that the discovery of such caveats and exceptions shouldn't inspire general skepticism about our scientific-induction presumption that there are elegant general laws which apply everywhere in the logical case – any more than discovering an unexpected new type of star should lead us to reject the project of astronomy (trying to learn laws that constrain the universe as a whole based on some finite portion of it we have access to) as impossible or unreliable.

[46] For example, such accounts would appeal to the logical possibility of satisfying the Peano Axioms which require the existence of infinitely many different numbers.

A critic might advance the following analogy: saying that elegant generalization from facts about finite and countable collections yields principles which accurately describe what is logically possible for some of the larger collections considered in pure mathematics is like saying that inference to the best explanation plus observations of birds in New Mexico explains our possession of true beliefs about birds in Canada as well. Presumably, in the ornithological case, we need to go gather more data in order to get many true beliefs about birds in Canada. But, in the mathematical case, we can't gather more data. Thus, our apparent possession of substantial true beliefs about what is logically possible for larger infinite collections remains mysterious on the story I have proposed.

I want to respond to this worry by accepting the analogy about birds above and saying that it fits the current state of human knowledge with regard to facts about the higher infinite rather well. Even in the case of birds, we can arrive at some true beliefs about birds in Canada just by inference to the best explanation from the facts about the birds in New Mexico. If we discovered tomorrow that some new island which had never yet been visited by explorers contained birds, I think we would reasonably expect many facts to carry over: any birds on that island would breathe oxygen, that they would have hollow bones etc. Our expectations about birds on this island would just be more sparse and less confident than our beliefs about birds in locations that we have observed.

But, this is just what happens with regard to our beliefs about logical possibility and large collections: as one moves from logical possibility facts concerning finite collections to those concerning countably infinite collections (like the natural numbers), and then uncountable collections (like the sets) our beliefs do get more sparse and less confident. For example, the continuum hypothesis[47] (CH) is a fairly simple statement involving sets of (relatively) small infinite size, yet it is known that (assuming ZFC is consistent) both the truth and the falsity of CH are compatible with ZFC. Our beliefs about what *large* infinite collections of objects and relations are logically possible are also frequently less confident than our beliefs about what finite and countable collections of objects are logically possible. Sociologically, mathematicians are frequently much more confident in their claims about numbers, sets of numbers and sets of sets of numbers than in the distinctive claims of set theory about what much larger patterns of mathematical objects would have to be like[48].

Thus, I think this last worry points to something that is an attractive feature rather than a flaw of the account at hand: it explains why we have relatively sparse beliefs about what's logically possible with respect to large collections, and hence relatively sparse beliefs about the corresponding facts concerning higher set theory.

However, this naturally raises the question of how much accuracy about higher set theory mathematics can be explained in the way I propose. Personally, I'm inclined to think that logical possibility facts are sufficiently uniform for the process of reflective equilibrium outlined above to account for our accurate recognition of the logical possibility of ZFC (and thus all the theorems of mainstream set theory). But someone who presumes less uniformity within logical possibility facts (and takes a more skeptical attitude to the higher reaches of set theory) may accept accept my solution to access worries concerning the mathematics they believe in (e.g. number

[47]The continuum hypothesis states that there are no sets whose cardinality is intermediate between the cardinality of the real numbers and that of the natural numbers. See [25] pg 176-186 for the proof that the continuum hypothesis is independent of the Zermelo-Fraenkel axioms.

[48]Think of choice vs. countable choice and disputes over large cardinal axioms.

theory), while denying that kind of uniformity we can expect from logical possibility is strong enough to get us to accurate principles about ZFC.

6. Contrast with Quine

I will conclude this paper by comparing my sample explanation for human accuracy about realist mathematical facts with the closest well-developed proposal in the literature, Quine's empiricist approach to mathematics[49].

I will argue that considering my proposal promises to answer access worries, in a way that considering Quine's does not. For my proposal can accommodate various blocking conditions and apparent features of our access to mathematics which no variant on Quine's core mechanism can[50]. To see this, note that my story differs from Quine's in three major ways.

First, where Quine's proposal takes dealings with concrete objects to push us to recognize the existence of *the particular mathematical structures* which we use in the sciences, my story takes dealings with concrete objects to push us to accept correct *general inference methods* which can be used to

[49]Modal Structualists like Hellman[23] and Shapiro[40] both gesture at the idea that something similar to Quine's approach could be used to explain knowledge of logical possibility. While their proposals only explain how we might come to accept mathematical structures indespensible to our best scientific theory, my proposal takes experience to correct our *general methods* of reasoning about logical possibility. As a result I avoid the Quinean problems about recreational mathematics noted below.
This Quinean starting point seems to have been incorporated by friends and foes of modal struturalism alike. For example in [32] Penelope Maddy argues that applications of mathematics can't explain our accuracy about set theory because everything we need to talk about has countable models (she is assuming physical theories are first order), as though experience had to correct our beliefs by directly forcing us to quantify over a structure rather than by motivating the acceptance of inference principles which imply the logical possibility of a range of structures. I hope this and other novelties of my proposal could be accepted by them as a friendly amendment by Hellman and Shapiro.

[50]I claim that (depending on how you want to describe the situation), either Quine's proposal fails to solve the access problem as the historical features it fails to account for always counted as blockers or succeeds in meeting some initial access problem only to fall victim to a secondary access problem about how we can have knowledge given these features. In contrast, I believe the reader will share my intuition that the ways in which my proposal is unrealistic don't create the impression that no explanation meeting these criteria is possible but, rather that sufficient scientific/historical diligence would surely yield some variant on my approach meeting these proposed blocking conditions.

derive the logical possibility of various mathematical structures. By endorsing this more indirect relationship between scientific and mathematical beliefs, it naturally avoids the 'problem of recreational mathematics' that besets Quine, i.e., it accounts for our knowledge of mathematical objects and structures which are scientifically useless. Similarly it accounts for mathematicians' tendency to learn abut scientifically useful mathematical objects *before* any scientific usefulness is discovered[51].

My story also makes good sense of the apparent cavalierness of both physicists and mathematicians with regard to positing new mathematical structures [52]. For one can say that (in such cases) mathematicians and physicists are usually already convinced of general methods of reasoning which let them derive the logical possibility of suitable structures (due to prior experiences and perhaps selection on an evolutionary time scale), and, on the views in the structuralist consensus, this is enough for them to correctly[53] use such a structure.

Second, where Quine's story appeals to *continuing indispensability* mine appeals to *past usefulness*. If (as Field argues in the case of Newtonian Mechanics[12] all quantification over mathematical structures in physics is ultimately dispensable, this would be a problem for Quine's empiricism but

[51] C.f. [18] on this historical point.

[52] As Justin Clark-Doane notes, physicists appear to make new mathematical postulates much more freely than they make new physical postulates – which seems odd on a Quinean picture where our acceptance of both types of objects is motivated by the same kind of inference to the best explanation[6]. Mathematicians seem equally cavalier about positing new objects in cases where there is reason to think the relevant structure is logically possible. As Julian Cole puts it, "Reflecting on my experiences as a research mathematician, three things stand out. First, the frequency and intellectual ease with which I endorsed existential pure mathematical statements and referred to mathematical entities. Second, the freedom I felt I had to introduce a new mathematical theory whose variables ranged over any mathematical entities I wished, provided it served a legitimate mathematical purpose. And third, the authority I felt I had to engage in both types of activities. Most mathematicians will recognize these features of their everyday mathematical lives."[9]

[53] More technically (remembering the case of fictionalism) this is enough for them to be as correct as any use of mathematics is taken to be.

not for my proposal. All that is necessary for my story to work is that recognizing the logical possibility or impossibility of various claimed patterns of relationships between concrete objects was practically useful at whatever time our dispositions to reason about logical possibility were formed.

Third, while Quine says mathematical knowledge is empirical, my explanatory story is entirely compatible with mathematical knowledge being a priori[54].

7. Conclusion

In this paper, I noted that Modal Structuralism allows us to transform the classic access problem for mathematics into an access problem for knowledge of logical possibility (and many other currently popular truth value realist philosophies of mathematics seem able to do the same). I then suggested that one can solve this residual access problem by noting the role which our general methods of reasoning about logical possibility can play in our attempts to predict and explain the behavior of concrete objects.

The above argument shows that access worries can be solved for at least one form of truth-value realism. I think it is also motivates significant optimism about whether other views in the structuralist consensus (that mere accuracy about coherence is enough to guide our mathematical posits) can solve their access problems. Those views may disagree about the underlying nature of mathematical claims. For instance some views in the consensus

[54]More specifically my explanation is compatible with mathematical knowledge being a priori on an ordinary foundationalist understanding of the a priori, which traces all a priori knowledge back to some basic principles and inferences which we can be warranted in making without justificatory appeal to anything else. If you think that *any* beliefs can qualify as basic a priori knowledge, beliefs directly produced by the application of correct general methods of reasoning about logical possibility which we find immediately compelling (and are perhaps even innately hardwired to find unquestionable) seem like an obvious candidate. I do not claim that my story it is compatible with the idea that mathematical knowledge is indubitable or strongly 'strongly a priori' in the sense of Fields "Recent Debates about the A Priori"[14]. But see Kitcher [26] chapter 1 for some independent reasons for doubting that mathematics is a priori in this sense.

such as Quantifier Variance are object-realists as well as truth-value realists while, others might spelling mathematical claims out in terms of fictions rather than logical possibility. None of this, however, prevents them from accepting that the mechanisms of correction I describe as an explanation for how creatures like us could have got reliable judgments about what is and isn't coherent.

I'd like to conclude by tentatively suggesting the following deeper picture of what is going on. Our body of mathematical theory seems to involve a mixture of insight into necessary a priori constraints on the behavior of all objects with mathematicians (oft-remarked on) apparent freedom to artistically choose which mathematical structures to talk in terms of[55].

Mathematics is as Quine puts it, 'black with fact and white with convention'[37] Using the notion of logical possibility lets us separate out a certain aspect of conventional choice in mathematics (which coherent package of mathematical structures to talk in terms of) leaves us with a less conventional subject matter which is a little more directly controllable and/or correctable by its applications.

Accordingly, breaking up the access problem into a question of explaining our accuracy about logical possibility and a free choice of which structures to talk in terms of is helpful in providing the kind of clear and intuitive model for how human beings could have acquired our current degree of accuracy about mathematics which is needed to truly satisfy (as opposed to merely silence) access worries.

[55]As Paul Lockheart puts it in his famous essay about math education, "in mathematics... things are what you want them to be. You have endless choices; there is no reality to get in your way. On the other hand, once you have made your choices (for example I might choose to make my triangle symmetrical, or not) then your new creations do what they do, whether you like it or not. This is the amazing thing about making imaginary patterns: they talk back! The triangle takes up a certain amount of its box, and I don't have any control over what that amount is. There is a number out there, maybe it's two-thirds, maybe it isn't, but I don't get to say what it is. I have to find out what it is."[31]

Appendix A. A More Formal Approach to Conditional Logical Possibility

I take the notion of conditional logical possibility to be primitive and intuitive. However, one can provide approximately correct truth conditions for sentences involving nested applications of subscripted $\Box$ and $\Diamond$ operators, in terms of the more familiar language of set theory with ur-elements.

First let us define a formal language $\mathcal{L}$, which I will call the language of logical possibility (though this language may be not able to express all meaningful claims involving logical possibility). Fix some infinite collection of variables and a collection of relation symbols, and define $\mathcal{L}$ to be the smallest language built from these variables using these relation symbols and equality closed under applications of the normal first order connectives, quantifiers, $\Box$ and $\Diamond$ (where the latter two operators can only be applied to sentences, so there is no quantifying in).

Specifically, if we ignore the possibility of sentences which demand something coherent but wouldn't have a model in the sets, (such as sentences which require the existence of proper class many objects) and take all quantifiers appearing outside a logical possibility operator to be implicitly restricted to some set sized domain of non-mathematical objects[56] we could say the following[57]:

[56]Our set theoretic approximation won't be able to adequately mimic all actual objects if there are 'more' actual objects than there are sets. Note that if you are an actualist about set theory, then the machinery of conditional logical possibility lets you describe structures strictly larger than the sets, e.g. adding one layer of sets on top of V.

[57]Note that if you are a potentialist about set theory in the sense of [36] and [23] these conditions do capture correct truth conditions for logical possibility but can't be used to *define* logical possibility on pain of circularity

Definition A formula ψ is true relative to a model $\mathscr{M}$ and an assignment ρ which takes the free variables in ψ to elements in the domain of $\mathscr{M}$[58] just if the following conditions obtain (note that only the last clause says something out of the ordinary):

- $\psi = R_n^k(x_1 \ldots x_k)$ and $\mathscr{M} \models R_n^k(\rho(x_1), \ldots, \rho(x_k))$.
- $\psi = x = y$ and $\rho(x) = \rho(y)$.
- $\psi = \neg\phi$ and ϕ is not true relative to $\mathscr{M}, \rho$.
- $\psi = \phi \wedge \psi$ and both ϕ and ψ are true relative to $\mathscr{M}, \rho$.
- $\psi = \phi \vee \psi$ and either ϕ or ψ are true relative to $\mathscr{M}, \rho$.
- $\psi = \exists x \phi(x)$ and there is an assignment ρ' which extends ρ by assigning a value to an additional variable v not in ϕ and $\phi[x/v]$ is true relative to $\mathscr{M}, \rho'$[59]
- $\psi = \Diamond_{R_1 \ldots R_n} \phi$ and there is another model $\mathscr{M}'$ which assigns the same tuples to the extensions of $R_1 \ldots R_n$ as $\mathscr{M}$ and $\mathscr{M}' \models \phi$.[60]

Set Theoretic Approximation: A sentence in the language of logical possibility is true *simpliciter* iff it is true relative to a set theoretic model whose domain consists of the actual objects (which the quantifiers in our special non-mathematical object language range over) and whose extensions for atomic relations reflects the actual extensions of these relations and the empty assignment function ρ.

Note that this definition gives statements lacking any necessity operators the same truth values as they have in the actual world.

[58]Specifically: a partial function ρ from the collection of variables in the language of logical possibility to objects in $\mathscr{M}$, such that the domain of ρ is finite and includes (at least) all free variables in ψ

[59]As usual $\phi[x/v]$ substitutes v for x everywhere where v occurs free in ϕ, and I am taking $\forall$ to abbreviate $\neg\exists\neg$

[60]As usual I am taking $\Box$ to abbreviate $\neg\Diamond\neg$

Appendix B. Translation of Mathematical Claims

To see how pure mathematical claims can be expressed in the language of logical possibility we offer a two step procedure. We first note (following Hellman [19]) that almost any (pure) mathematical claim ψ can be given a nominalistic paraphrase involving both logical possibility and second order logic. We will then transform this second order nominalistic paraphrase into a pure logical possibility claim.

The starting nominalistic paraphrase is formulated in terms of a categorical[61] second order description D of a mathematical structure (the mathematical structure ψ talks about) with domain M under mathematical relations $S_1 \ldots S_k$. We assume all first order quantifiers in D and ψ are *explicitly* (rather than implicitly as is the mathematical convention) restricted to a predicate M representing the domain of the mathematical structure under investigation[62] and no logical possibility operators appear in D or ψ).

We can transform this second order description of a mathematical structure D into a suitable $D_\Diamond$ by applying a translation procedure t which we

[61]Non-categorical descriptions like the axioms of group theory can be handled as well by replacing the hypothetical form below with the slightly more complex form so as to both assert the possibility of some structure satisfying the description and assert that any structure satisfying the description renders ψ true.

[62]That is, we can assume all quantifiers are of either the form $(\exists x)(M(x) \wedge \phi)$ or $(\forall x)(M(x) \rightarrow \phi)$. Note that this has the effect of replacing any second order collection X appearing in D or ϕ with $X \cap M$ since facts about membership in X can only be queried on M.

can recursively define as follows (with $t = t_{(S_1 \ldots Sk)}$).

$$t_{(R_1..R_n)}(\exists P\phi) = \Diamond_{M,R_1\ldots R_n} t_{(R_1\ldots R_{n+1})}(\phi[P/R_{n+1}])]$$

$$t_{(R_1..R_n)}(\forall P\phi) = \Box_{M,R_1\ldots R_n} t_{(R_1\ldots R_{n+1})}(\phi[P/R_{n+1}])]$$

$$t_{(R_1..R_n)}(\neg\phi) = \neg t_{(R_1..R_n)}(\phi)$$

$$t_{(R_1..R_n)}(\phi \wedge \psi) = t_{(R_1..R_n)}(\phi) \wedge t_{(R_1..R_n)}(\psi)$$

$$t_{(R_1..R_n)}(\phi \vee \psi) = t_{(R_1..R_n)}(\phi) \vee t_{(R_1..R_n)}(\psi)$$

$$t_{(R_1..R_n)}(\exists x\phi) = (\exists x)[t_{(R_1..R_n)}(\phi)]$$

$$t_{(R_1..R_n)}(\forall x\phi) = (\forall x)[t_{(R_1..R_n)}(\phi)]$$

$$t_{(R_1..R_n)}(R_k(x_1, ..x_m)) = R_k(x_1, ..x_m)$$

$$t_{(R_1..R_n)}(x_1 = x_2) = x_1 = x_2$$

So, for example, suppose ϕ is the following second order sentence about the natural numbers under $S, +$ and $\times$, with quantifiers restricted to objects satisfying M:

$\forall X \exists Y \forall z[\mathbb{N}(z) \wedge P_1(z) \rightarrow (\exists z')(\mathbb{N}(z') \wedge S(z', z) \wedge Y(z'))]$

then $\phi_\Diamond$ has the form

$\Box_{\mathbb{N},S,+,\times} \Diamond_{\mathbb{N},S,+,\times,P_1}[\mathbb{N}(z)P_1(z) \rightarrow (\exists z')(\mathbb{N}(z') \wedge S(z', z) \wedge P_2(z'))]$

where P_1 and P_2 are some non-mathematical predicates, e.g., 'cat' and 'dog'. Thus we can write a suitable potentalist translation for statements about the mathematical structure described by D as follows $\Diamond(D_\Diamond) \wedge \Box(D_\Diamond \rightarrow \phi_\Diamond)$[63].

[63]Recall that we can use Putnam's trick of replacing mathematical relations with non-mathematical vocabulary of the same arity inside of the $\Diamond$ context to purge any use of mathematical vocabulary from this paraphrase.

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