PHYSICAL POSSIBILITY AND DETERMINATE NUMBER THEORY

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Abstract. In this paper I will propose a new answer to classic model-theoretic worries about how we can grasp a definite natural number concept. I review some problems for existing 'indefinite extensibility' proposals, which answer this worry by saying that our expectations about the relationship between number theory and all possible 'logic preserving' expansions of our language rule out non-standard interpretations of number theory. I will argue that one can avoid these problems by instead appealing to our expectations about the relationship between mathematics and *physical (or metaphysical) possibility* to rule out non-standard models.

1. Introduction

Do we have determinate (categorical) conceptions of mathematical structures like the natural numbers? It intuitively seems like we do. But there's a classic worry (inspired by considering non-standard models of first order Peano Arithmetic) that this is not so, which Putnam raised in [8] and Warren and Waxman have recently emphasized in [9].

In this paper I will propose a new answer to this challenge. First I will review some problems for existing proposals, including 'indefinite extensibility' based responses to the model theoretic challenges, which maintain that certain expectations about the relationship between number theory and all possible 'logic preserving' expansions of our language suffice to rule out non-standard interpretations of number theory. Then I argue that we can cleanly avoid these problems by claiming that our expectations about the relationship between mathematics and *physical (or metaphysical) possibility* to suffice to rule out non-standard models.

Specifically, I will argue that if one grants that we can (somehow!) latch on to a definite notion of physical or metaphysical possibility, then this fact together with our dispositions to assert that it would be physically impossible for certain concrete first order definable properties to apply counter-inductively (i.e. to 0 and the successor of 0 but not to all numbers) suffices to rule out non-standard interpretations of our number talk.

Admittedly, it would be very strange if our ability to think about the natural numbers depended on our ability to grasp the concept of physical or metaphysical possibility. So, I should stress, I don't claim the above-mentioned facts about us constitute our only grip on the intended structure of the natural numbers, or a necessary condition for thinking about this structure. I (only) claim that they provide a *sufficient* condition for grasping a determinate number concept.

However, I think that recognizing this sufficient condition is very important because of the connection it revels between realism about physical possibility and realism about mathematics. By highlighting this scenic side-route to having a definite number theoretic concepts (while allowing that the royal road lies elsewhere) I hope to convince metaphysicians who happily accept robust facts about metaphysical or physical possibility but are agnostic about mathematical realism that resources they are already committed to accepting suffice to answer Putnam's model theoretic challenges – so they shouldn't be deterred from accepting truthvalue realism about mathematics on these grounds.

2. Putnam's Model-Theoretic Challenge

Let me begin by laying out Putnam's model-theoretic challenge, as it applies to our conception of the natural numbers.

The standard first order axioms of arithmetic (PA) plausibly articulates part of our concept of numbers (in delineating restrictions on how the symbols $\mathbb{N}, S, +, *, <$ can relate). However, these axioms can also be satisfied by non-standard models with a different structure and may change the truth-value of some arithmetic sentences. For instance, PA requires that every number besides 0 both have and be a successor, but this leaves open the possibility of non-standard interpretations which (under $<$) look like the following (where each additional $*$ indicates a disjoint copy of the integers):

$$0, 1, 2, 3, \ldots, -2^*, -1^*, 0^*, 1^*, 2^*, 3^*, \ldots, -2^{**}, -1^{**}, 0^{**}, 1^{**}, 2^{**}, 3^{**}, \ldots$$

The resulting structure looks like a copy of the natural numbers followed by two copies of the integers. Note that by ensuring there is no least 'infinite' number such a structure can satisfy the requirement about every element besides 0 both being and having a successor.

This alone isn't enough to create a non-standard model of PA[1]. However, if we instead consider the structure consisting of a copy of the natural numbers followed by infinitely many copies of the integers *densely ordered* (i.e., the resulting structure has the form $\mathbb{N} + (\mathbb{Z}) \cdot Q$ where $\mathbb{Z}$ is just the integers and Q is the rationals)[5], then there is a way for the relations $+, *, <$ to apply so that all so that all the Peano axioms are satisfied – including all instances of the first order induction schema[2].

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[1]Note that we've only said that the basic first order Peano axioms about S applies are satisfied in the structure above, not that any of the other ones (i.e. those for $+, \cdot$ and $<$) are.

[2]That is, all sentences of the following form in the language of arithmetic $\phi(0) \wedge (\forall n)[\phi(n) \rightarrow \phi(S(n))] \rightarrow (\forall m)\phi(m)$.

However, such non-standard models of PA don't satisfy the full second order induction axiom, $(\forall X)\,[(X(0) \wedge (\forall n)(X(n) \to X(S(n)))) \to (\forall m)X(m)]$. Indeed, the set X consisting of just the standard integers (i.e., the objects represented by '$1, 2, 3 \cdots$' in the example above) satisfies $(\forall n)[X(n) \to X(S(n))]$ but doesn't satisfy $(\forall n)X(n)$.

In view of the existence of such non-standard models, one can ask (as Putnam does) the following question. Do we really have a definite concept of 'the structure of the natural numbers' which is not satisfied by any non-standard interpretations? What can such a concept consist of? What is it about us which (perhaps together with facts about the world, intrinsic eligibility etc.) lets us our words like "number" and "plus" take on meanings which rule out such non-standard models? For reasons I won't discuss here, Putnam takes our ability to give standard meanings to the first order logical vocabulary for granted in his challenge. I will follow him in doing so.

With this in mind, we can dramatize Putnam's challenge as follows. Imagine some all-knowing interpreter who is dedicated to interpreting our talk about the natural numbers in some unintended fashion. This malign [mischevious] interpreter has full access to ordinary determinate mathematics and uses that knowledge to construct non-standard models for our talk of the natural numbers to refer to.

Can we cite plausible constraints which our mischievous interpreter must honor which prevent him from giving an unintended interpretation? Note that classic results in mathematical logic [4] tell us that no further mathematical specification (i.e., extending PA or embedding the numbers in a larger structure) could provide such a constraint. And note that people often invoke our causal contact with objects like rabbits and electrons as part of an answer to Putnam's more general model-theoretic challenge (which applies to scientific and everyday objects just as much as mathematical abstracta). But plausibly (even if it works generally) this appeal to causal contact isn't available for answering Putnam's challenge with respect to the natural numbers.

If we can give no satisfying answer to Putnam's challenge, then, perhaps, we must allow that our conception of the structure of the natural numbers is vague and allows for a range of acceptable precicifications (corresponding to different structures satisfying the Peano axioms, much like the range of acceptable precisifications of 'bald()' and 'heap()'). As Field[2] has emphasized, this view would still allow us to use classical logic when reasoning about the natural numbers (because, e.g., formulas of the form '$P \vee \neg P$' will be true on all acceptable precicifications). But admitting our conception of the natural numbers is vague involves significant bullet biting with regard to the idea that all statements of arithmetic (even ones we can't decide) have definite truth-values – as the most common way of ensuring such definite truth values is through reference (up to isomorphism) to the natural numbers [3].

So I will try to resist this unhappy conclusion, by providing an answer to Putnam's challenge here.

3. Background and Contrast with Open-endedness approaches

Let me begin by quickly reviewing the two closest proposals in the existing literature.

3.1. **The Language Expansion Approach.** It would be very easy to answer the challenge above if one could take for granted our ability to definitely

[3] In principle, one could take there to be non-isomorphic acceptable models, but only ones that agree on all sentences in the language of arithmetic. However, any attempt to secure reference to the theory of the natural numbers (without assuming we can reference (up to isomorphism) the natural numbers themselves) would seem to run into exactly the same kind of problems securing reference to the natural numbers involved in the first place.

grasp second order quantification. In this case, one could use our acceptance of *second order* Peano Arithmetic, (the version of Peano Arithmetic which replaces the first order induction axiom schema with the second order induction axiom repeated below) to explain what is wrong with non-standard models of the natural numbers. For, as noted above, the non-standard models of PA don't satisfy full second order induction.

In [7] [6] Parsons and McGee have offered an answer to the Putnamian challenge centering on what McGee calls **openendedness**: the idea that we expect all instances of the first order induction axiom schema to continue holding true in any "logic preserving" extension of our language.

McGee argues (roughly[4]) as follows. Suppose (for contradiction) that some nonstandard model M provided an acceptable interpretation of our terms 'natural number', 'successor' etc. Then there could (in some sense) be a god who is able to point to the non-standard model and introduce a term "smee" which applied counter-inductively to this non-standard model (i.e., smee applies to 0, and $smee(n) \implies smee(S(n))$, but smee doesn't apply to every 'natural number'). If we met such a god then we could (logic-preservingly) extend our language by taking the term 'smee' from their language and adding it to ours. In such a case, we would still expect the induction axiom to hold for formulas involving smee which we got from talking to this god. Therefore, interpreting us to mean a nonstandard model is unacceptable because it would fail to satisfy induction in this extended language.

This strategy has drawn a variety of criticisms. First, it might well be metaphysically impossible for a god to introduce a term like smee. For instance, It's not clear how the god could refer sufficiently definitely to some

[4]McGee's actual proposal is somewhat more complicated in ways that I claim don't effect any of the criticisms discussed here. See [6] pgs 56-68.

proper initial segment of our non-standard model. What can the god do to secure reference in a way we cannot? Are we to imagine a metaphysically impossible scenario where they fly into the realm of abstract objects and point one by one to each of the infinitely many elements in the initial segment? Maybe we should imagine they perform some supertask with physical objects that pins down this initial segment[5]. Perhaps this is close to what Field and in mind when he expressed a worry about something like: why can't we just say that we secure definite reference by whatever we are imagining the god to do to secure her reference? in [2][6].

McGee seems happy to accept the metaphysical impossibility of the scenarios he envisages and instead appeals to a the idea that we are committed to the first order induction schema being true in all *logically* 'possible' extensions of our language. He writes:

> To say what individuals and classes of individuals the rules of our language permit us to name is easy: we are permitted to name anything at all. For any collection of individuals K there is a logically possible world-though perhaps not a

[5]If the god just introduces standard explicit definitions this doesn't seem to increase the expressive power. Maybe we are supposed to imagine them introducing a truth predicate for our language and then another on and on. But there's much debate over what happens with such truth predicates. Relatedly, the usual answer to Kaplan's paradox (if there are α worlds then there are 2^α possible propositions so, e.g., it can't be the case that for each proposition there is a distinct possible world at which only that proposition is expressed) provides strong reason to think many 'combinatorially possible' ways a language could work are actually *not* metaphysically possible for anyone to have.

[6]Field writes, "...how can adopting McGee's rich view of schemas help secure determinacy? That view of schemas merely allows me to add an instance of induction whenever I add new vocabulary. But the relevant vocabulary for McGee's argument would seem to be 'standard natural number', and we've already seen that that is no help. Of course, it's true that if I could add a predicate that by some magic has as its determinate extension the genuine natural numbers, then I will be in a position to have determinately singled out the genuine natural numbers. That's a tautology, and has nothing to do with whether I extend the induction schema to this magical predicate. But if you think that we might someday have such magic at our disposal, you might as well think we have the magic at our disposal now; and again, it won't depend on schematic induction. So the only possible relevance of schematic induction is to allow you to carry postulated future magic over to the present; and future magic is no less mysterious than present magic."

> theologically possible world-in which our practices in using English are just what they are in the actual world and in which K is the extension of the open sentence 'x is blessed by God'. So the rules of our language permit the language to contain an open sentence whose extension is K[6].

However, one might worry that our dispositions in metaphysically impossible scenarios like the above are not clearly enough understood to be invoked in this context. Additionally, one might worry that such counter-possible conditionals don't so much explain our ability to determinately refer to the natural numbers as package the intuition that we do.

One might also object that availing ourselves of the space of all logically possible extensions of our language to explain how we have a determinate conception of the natural numbers is question begging. We wouldn't accept an explanation that presumed we have a determinate conception of second order quantification and it's not clear that considering all logically possible linguistic extensions is materially different. And if we can somehow intend that the induction schema remain true in languages corresponding to all possible ways of choosing a subset of individuals for a predicate to apply to, why can't we use the same faculty to directly expect that our second order quantifiers range over every possible subset. Indeed, one might doubt that we even have a definite conception of a logic preserving extensions of our language[7] at all.

3.2. **Appealing to the Actual Structure of Time.** Hartry Field (rather ambivalently) proposes an alternative account, on which he argues that *if*

[7]Hartry Field also raises a worry about whether (in the scenario above) the god's term can be permissibly added to our language which I'm not sure that I buy. However, whether or not the objection is ultimately persuasive, clearly no analog applies to the proposal I will offer below.

time forms a genuine ω sequence[8] (i.e., time has infinite duration and there are only a finite number of seconds between any two times) then our belief that this is true can be used to rule out nonstandard interpretations of our number talk (given standard interpretations of our temporal and event talks). I don't think this proposal works, as we treat the assumption that time forms a genuine ω sequence as a contingent hypothesis and not a conceptual truth constraining what we mean by 'the natural numbers'. Thus, I don't see why acceptable interpretation of our language must make this hypothesis about the numbers true. For example, if I believed that the number of gumballs in the jar is 70, this belief presumably wouldn't commit a mischievous interpreter to interpret the concept 'natural number' in such a way as to make this statement come out true. Moreover, even if one accepts Field's argument it only claims to establish a determinate reference (up to isomorphism) of the natural numbers *if* time forms a genuine ω sequence.

4. My Proposal

4.1. **Expectations about physical possibility instead.** I will now present a different answer to Putnam's challenge, which avoids all the difficulties above. The key idea will be that the interaction of physical possibility and mathematical facts provide a new route to excluding non-standard models. More precisely, I will argue that if our mischievous interpreter is well behaved (in the sense of satisfying the following conditions) then they will be unable to trick us into talking about a non-standard model of the natural numbers.

Say an interpreter is well behaved iff

[8]An ω sequence refers to a collection of elements which, under some relation $<$, has the same structure as the intended model of the natural numbers, i.e., is comprised of a first element, the successor of that element and so forth. Note that the claim time forms an ω sequence (assuming it is linearly ordered) is equivalent to the claim that if we start marking off one second intervals at any point those marks form an ω sequence.

- They select a single model as the referent of our concept 'natural number' at all physically possible worlds.
- They cannot tamper with extension of the following non-mathematical vocabulary: 'coinflip' 'heads' 'temporally after' at any of these physically/metaphysically possible worlds.
- They give the usual meaning to logical vocabulary and physical (or metaphysical) necessity operator, e.g., the existential quantifier and the physical necessity operator $\Box_p$ must contribute to truth conditions in the usual fashion. However, the interpreter is free to choose any model for the natural numbers by selecting a domain (the objects masquerading as the natural numbers), a 'natural number' for the constant 0 from that domain and functions $S, +, *$ on the 'natural numbers'[9].
- They must make all statements which we are willing to endorse as conceptually required by our grasp of the natural numbers (such as the Peano axioms) come out true. Note that this requirement extends beyond purely mathematical vocabulary by extending the induction schema to encompass any total (definitely true or false for every input) property on the 'natural numbers' describable in our current language[10]. Thus, if $Q(n)$ abbreviates "the n-th coinflip is heads,", there is a k-th coinflip for each 'natural number' k then from $Q(0) \wedge (\forall n)\,[Q(n) \implies Q(S(n))]$ we can infer $(\forall n) Q(n)$
- They must vindicate the conceptual truths relating the numbers and the practice of counting a sequence of events in time (specifically it suffices to vindicate those truths specified in section 4.2). In other

[9] We will also make use of the relation $<$ but regard $x < y$ merely as an abbreviation for $(\exists z)(x + S(z) = y)$.

[10] More restrictively, it suffices to augment the mathematical vocabulary with the terms in 4.1 and 4.1

> words the interpreter must allow us to match up a sequence of events to natural numbers in a way that obeys the usual rules for counting (first event corresponds to 0, the next event always corresponds to 1 plus it's predecessor etc..) .

Faced with a well behaved, but still infinitely devious, interpreter our strategy will be to identify a sentence ϕ about the physically possibility of a sequence of coin flips, which our opponent is bound to interpret as true but would be false if the referent of natural number was non-standard. Peeling back the evocative metaphor of our mischevious interpreter, it is this sentence which provides us a determinant grip on the structure of the natural numbers. Taking ϕ to be partially constitutive of the concept of natural number this ensures that its reference is the intended model.

To ensure that ϕ has the correct truth conditions we will need the following uncontroversial assumption about physical possibility:

> Infinite Random Sequence (IRS): It is physically possible to have a series of independent *objectively* random events linearly ordered in time[11] with two possible outcomes ('heads' and 'tails') having a first event but no final event. Furthermore, every event in the series has a temporal successor, i.e., for any event x there is some other event y occurring after x such that no event z occurs between x and y.

Informally, one can think of the events whose possibility IRS asserts as being like the ticks of an indestructible watch which never needs repair or

[11]That is for any distinct events x, y in the series either x occurs before y or y occurs before x. Moreover, from the point of view of relativistic physics, the measurements are separated by time-like intervals (x is in the future lightcone of y or vice versa) so all observers agree on their order. Given these constraints it is safe to simply work relative to some fixed inertial reference frame and ignore relativistic complications for the remainder of the paper.

winding. There is a first tick, each tick is followed by a unique next tick and there is no tick after which the watch breaks down.

To motivate accepting this principle note that it is only asserting that it is physically possible to repeatedly perform (independent) textbook spin measurements on an electron[12][1] (or some other equivalent process) and that the laws of physics don't rule out time continuing infinitely into the future (though possibly having non-standard 'length')[13]. I will abstract away from the details of the measurement and simply refer to it as a 'coinflip' and the two outcomes as 'heads' and 'tails.'

4.2. **Pinning Down The Natural Numbers.** I propose to answer Putnam's challenge by invoking the fact that we expect the induction schema to hold at all possible worlds. Specifically I will note that if IRS there is a physically possible world at which the coinflips that come up heads form a genuine ω sequence. And, using this fact, I will argue that a certain natural language property applies counter-inductively to any non-standard model of the 'natural numbers' in this world[14].

To see how the details work, note that our current mathematical language allows us to use natural numbers to talk about events taking place in time such as 'the 4th U.S. President' or 'the 37th successful rickrolling'. This practice of talking about the nth coinflip presumably includes accepting principles like, 'if no coinflip occurred before x, then x is the 0th coinflip.' I take the following such principles to be conceptual truths regarding counting (temporal) sequences of events using the natural numbers[15], where

[12] That is perform a spin measurement along the x-axis on an electron whose spin has just been measured (and thus collapsed) along the y-axis. Thanks to REDACTED for suggesting these details.

[13] We will see that, ultimately, the use of objective randomness is just a way to establish it would be physically possible for there to be a temporal ω sequence of of objects satisfying some property (having a determinate extension) in our current language.

[14] That is containing 0, closed under successor but not to all the 'numbers'

[15] C.f. [3].

$coinflip(x)$ denotes x is a coinflip, $countflip(n,x)$ denotes x is the n-th coinflip, $heads(x)$ denotes that coinflip x has the heads outcome and $before(x,y)$ denotes that the coinflip x occurs temporally prior to coinflip y.

- An object x is the 0th coinflip, i.e., $countflip(0,x)$ iff x is a coinflip and all other coinflips happen after x. $(\forall x)[countflip(1,x) \leftrightarrow coinflip(x) \wedge (\forall y)(countflip(y) \rightarrow before(x,y) \vee x=y))]$
- If x is the nth coinflip, then y is the $S(n)th$ coinflip iff y occurs after x and no other coinflip occurs between x and y. That is,

$$(\forall n,x,y)(countflip(n,x) \rightarrow$$

$$[(countflip(S(n),y) \leftrightarrow coinflip(y) \wedge before(x,y) \wedge (\forall z)\neg\,(coinflip(z) \wedge before(x,z) \wedge before(z,y))]$$

- Only coinflips can be the nth coinflip, i.e., $(\forall x)(\exists n)(countflip(n,x) \rightarrow coinflip(x))$
- No two distinct numbers correspond to the same coin flip. $(\forall n)(\forall m)[coinflip(n,x) \wedge coinflip(m,x) \rightarrow m=n]$

Together with IRS the above conceptual truths regarding counting ensure[16] that (at some physically possible world) our current vocabulary lets us pick out a counter-inductive[17] collection of numbers (thereby witnessing that the restrictions on our interpreter should have prevented that choice of non-standard model) .

[16]Note that in many physically possible situations there will be a 'number' n such that these analyticities plus the facts about how $countflip()$, $coinflip()$ and $before()$ apply insure that there is no nth coinflip for certain values of n. For example, if no coinflips take place after the nth coinflip there will be no n+1th coinflip. Even in worlds whose possibility is asserted by IRS it might be that there are only standard temporal durations, e.g., n-seconds after only makes sense for standard integers n, in which case those worlds wouldn't have any n-th coinflip where n is non-standard.

[17]These conceptual truths do not necessarily uniquely determine which elements n in the nonstandard model will be interpreted to satisfy '$(\exists x)coinflip(x,n) \wedge heads(x)$ but they insure that all such interpretations will be counter-inductive.

By IRS there is a physically possible world w where infinitely many coinflips (linearly ordered by temporally before) take place and all and only the initial ω sequence of these coinflips come up heads. We claim that the above conceptual truths force along with the constraints on our mischievous interpreter ensures she takes $P(n) \stackrel{\text{def}}{=} (\exists x)(countflip(x,n) \wedge heads(x))$ to hold for just those n in the standard initial segment of the nonstandard referent of the natural numbers. As only the initial ω sequence of coinflips land heads $P(n)$ can't hold outside of the standard initial segment. Since IRS ensures that there is an first coinflip and each coinflip is followed by an earliest subsequent coinflip refzero-flip and 4.2 guarantee that every 'standard' coinflip is counted by a standard number ensuring that $P(n)$ holds for every element in the standard initial segment. Therefore, induction fails at this physically possible world for the property $P(n)$ (expressed in terms of determinate concepts in our current language as evaluated at that world). Consequently the induction schema as applied to the above property fails to hold with physical necessity if natural number has a non-standard interpretation.

Finally note that exactly the same argument would work if we replaced appeal to a definite notion physical possibility $\Box_p$ with appeal to a definite notion of metaphysical possibility $\Box_m$.

Also, in contrast to previous approaches like that of Field, my proposal to secure determinate reference doesn't rely on any suspect notions like 'isomorphic to' or $1-1$ function[18]. For, if we are worried about determinate reference for the concept of natural number surely the more abstract concept of function on the natural numbers is, itself, also at issue.

[18]Instead, I rely on conceptual truths about the natural numbers and their role in counting.

5. Conclusion

In this paper I have argued that we can appeal to expected relationships between mathematical facts and physical or metaphysical possibility to rule out non-standard models of our number theoretic talk. I have also reviewed some worries for previous 'indefinite extensibility' based accounts of our ability to grasp a fully definite concept of the intended structure of the numbers, and noted that this approach avoids them.

Let me close on a note of humility by reminding the reader of the two caveats from the introduction. First, I admit that any philosopher of mathematics who doesn't think there are determinate right answers to all questions in number theory will be inclined to doubt our determinate grip on physical and metaphysical possibility which my response assumes. Rather, my aim is merely to argue that realism about physical and/or metaphysical possibility creates a lot more pressure to be truth-value realist about mathematics than many metaphysicians realize. Second, it would be strange if our possession of a definite conception of the natural numbers depended on our beliefs about physical (or metaphysical) possibility. Thus, I suspect another kind of answer to Putnam's challenge must be possible.

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