The Mathematical Nominalist's Real Problem With Physical Magnitudes

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Abstract

A key reason for thinking mathematical nominalists can't answer the Quinean indispensability argument concerns difficulties nominalistically paraphrasing physical magnitude statements. In this paper, I argue that nominalists who accept certain notions from the literature on potentialist set theory can avoid these difficulties by deploying two cheap tricks. Doing this lets us answer Quinean and Explanatory indispensability worries associated with physical magnitude statements.

1 Introduction

Indispensability arguments for mathematical Platonism maintain that (in one way or another) we cannot adequately make sense of our current scientific knowledge without accepting the existence of mathematical objects. The classic (Quinean) indispensability argument holds that we need to quantify over mathematical objects to *literally state* our best scientific theories, and this commits us to the existence of such objects. And explanatory indispensability arguments[2] point out that mathematical facts do the heavy lifting in certain scientific explanations, and maintain that mathematical objects are needed to *best explain* certain scientific facts.

A key source of worry about nominalists' ability to answer classic indispensability worries [23, 10, 17, 8] directly (i.e. by providing a paraphrase) concerns physical magnitude statements (i.e., statements about lengths, temperatures and the like). When formalizing a theory like Newton's law of gravity, the Platonist can appeal to a length function, which pairs each spatial path with its length-in-meters (a certain real number). And a nominalist paraphrase must simulate or replace such talk of a length function where it appears. Yet there are certain reasons (beginning with Putnam's influential counting argument[23] and continued with what I'll call a 'sparse magnitudes' problem[8, 11]) for fearing the nominalist can't do this. In this paper, I'll argue the mathematical nominalist can plausibly answer both classic Quinean and explanatory indispensability worries about physical magnitude statements, provided they accept certain logical machinery (independently motivated by work on potentialist set theory) and substantivalism about space (as Putnam and Hellman do).

In §2 I'll review basic indispensability worries and introduce the relevant modal notion (a kind of logical possibility). In §3 I'll present a basic modal if-thenist nominalization strategy[1]. I'll argue that (where this paraphrase strategy can be applied) it promises to address both classic and explanatory indispensability worries (and improve on Field[10] and Rizza[25]'s attempted paraphrases). In §4 I'll review how physical magnitude statements pose a continuing problem for applying this theory (and nominalist paraphrase in general). And in §5 I'll propose a solution to this problem via adding two formal 'cheap tricks' to the basic paraphrase strategy above.

Thus, I'll argue that (at least basic forms of) classic Quinean and explanatory indispensability worries about physical magnitude statements can plausibly be answered[2]. Admittedly, that the paraphrases I advocate won't be helpful to every nominalist. For example, philosophers who reject mathematical nominalism as part of a general physicalist project will probably reject my key notion of logical possibility as insufficiently physical. However, as Putnam notes[22], in many contexts we can seemingly equally well take either a modal or a Platonistic perspective on pure mathematics, and certain puzzles (concerning the

[1] c.f. [17, 4]

[2] Early versions of some key ideas from this paper are proposed as part of a larger technical project in [4].

Burali-Forti paradox and the height of the hierarchy of sets) appear to favor a modal approach to pure higher set theory [17, 19]. In this paper, I aim to clarify whether Quinean and/or explanatory indispensability arguments block taking a similarly modal perspective on mathematics as a whole.

2 Background and Motivation

2.1 Indispensability Arguments

The classic Quinean indispensability argument goes roughly like this[24]. We can't literally state our best scientific theories (in a logically regimented language) without quantifying over mathematical objects. And we should accept that all objects quantified over in stating our best scientific theories exist. So we should believe in mathematical objects.

Accordingly, Quine's indispensability argument challenges the nominalist to state a formalized version of their best total theory without quantifying over objects (like the numbers) they don't believe in. To make the logical regimentation demand at issue more concrete, I'll consider whether nominalists can adequately paraphrase an apparently Platonistic scientific theory S they accept, by producing a nominalistic paraphrase T(S) which *the Platonist must regard as* true at exactly those metaphysically possible worlds where S is true.

Some nominalists, like Hartry Field, have answered this challenge head on, by rewriting scientific theories to avoid quantification over mathematical objects[10]. Others have rejected this demand for literal statement [6, 1, 27]. Drawing on scientists' use of idealized models and known falsehoods, like talk of infinitely deep oceans or continuous population functions, they say that it's OK if we can only evoke the scientific claims we believe by engaging in a fiction/pretense and saying things that are (literally) false. According to the latter group of nominalists, we can unproblematically quantify over mathematical objects in communicating our best scientific theories, even though no mathematical objects exist.

The Explanatory Indispensability argument strikes back at both kinds of nominalists by suggesting mathematical objects are needed to **best explain** the data accounted for by our scientific theories. This argument suggests that even if we don't need to accept all objects quantified over in communicating our best scientific theories (or we can cook up a very, long ugly nominalistic paraphrase that technically implies the same constraints on non-mathematical reality), the existence of mathematical objects is required to best explain the data which motivates our some of our scientific theories. So we have an inference to the best explanation for the existence of mathematical objects.

The Stanford Encyclopedia of Philosophy gives the following example of a case where mathematical objects have been argued to be explanatorily indispensable[7]:

> [the] poster child for [arguments for the] explanatory indispensability of mathematical objects is Baker's Magicadas explanation.
>
> North American Magicadas are found to have life cycles of 13 or 17 years. It is proposed by some biologists that there is an evolutionary advantage in having such prime-numbered life cycles. Prime-numbered life cycles mean that the Magicadas avoid competition, potential predators, and hybridization. The idea is quite simple: because prime numbers have no non-trivial factors, there are very few other life cycles that can be synchronized with a prime-numbered life cycle. The Magicadas thus have an effective avoidance strategy that, under certain conditions, will be selected for. While the explanation being advanced involves biology (e.g., evolutionary theory, theories of competition and predation), a crucial part of the explana-

tion comes from number theory, namely, the fundamental fact about prime numbers.

And examples of cases where mathematical objects are putatively indispensable involving physical magnitude statements (like the claim that honey combs have a certain shape because this maximizes the volume-to-side area ratio, given certain constraints) can also be given.

In this paper, I'll argue that applying certain formal 'cheap tricks' to the kind of modal if-thenist paraphrases discussed in (c.f. Hellman in [17]) lets us address classic Quinean indispensability worries about nominalizing scientific theories involving physical magnitudes like length and charge. I'll argue these nominalist paraphrases are explanatorily at least as good as (and perhaps better than), corresponding Platonist explanations. So we can use them to answer associated explanatory indispensability worries as well.

2.2 Conditional Logical Possibility

Let me now introduce the key modal notion (independently motivated by work on potentialist set theory) to be used in nominalistic paraphrase strategy. For simplicity, I will write my paraphrases using the conditional logical possibility operator from [4]. However, my proposal can be easily reformulated using the logical machinery from other versions of potentialist set theory (e.g., Hellman's combination of a second-order function and relation quantifiers and a plain logical possibility operator in [18])[3].

To introduce the notion of conditional logical possibility, first note that there's independent reason for accepting a primitive notion of logical possibility

[3]We could possibly do the same work by coding all nominalistically acceptable objects and relations satisfying the definable supervenience conditions below with sets and then using the notions of interpretational possibility, the logically necessary essences of sets, and plural logic Linnebo uses to develop potentialist set theory in [20].

($\Diamond$) interdefinable with logical entailment[4]. When considering logical possibility in this sense, we ask what is possible while ignoring all constraints on the total size of the domain, and considering all possible ways of choosing n-tuples from this domain as extensions for various relations. When evaluating conditional logical possibility ($\Diamond_{R_1 \ldots R_n}$) we do almost the same, but hold fixed (structural facts about) how the subscripted relations $R_1 \ldots R_n$ apply.

To better motivate this idea, consider situations where there are more cats than baskets. In such cases, I would say that it's logically impossible (holding fixed structural facts about how cathood and baskethood actually apply) that each cat is sleeping in a different basket. I will write the latter claim as follows.

$\neg\Diamond_{cat,basket}$ [Each cat is sleeping in a basket and no two cats are sleeping in the same basket.]

We can also nest logical possibility operators, to discuss whether it's logically possible for some relations $R_1, ...R_n$ to apply a way that makes some target state of affairs logically possible/necessary (given the structural facts about how these relations $R_1, ...R_n$ apply).

For example, it's logically possible that there are three cats and two baskets. So, it's logically possible for 'cat' and 'basket' to apply in a way that makes it logically impossible (given the structural facts about how cathood and baskethood apply) that each cat is sleeping in a different basket. And we can write that claim as follows.

$\Diamond\neg\Diamond_{cat,basket}$ [Each cat is sleeping in a basket and no two cats are sleeping in the same basket.]

[4] I follow [12] in taking the $\Diamond$ of logical possibility as a primitive modal notion (that's a logical operator).

Admittedly there's a fruitful tradition of identifying logical possibility with having a set theoretic model for various mathematical purposes (and validity with not having a countermodel). However, there are independent reasons[15, 16, 5, 9, 14] for thinking we have prior grasp on the notion of logical possibility.

Also, one might feel (with Boolos) that, "one really should not lose the sense that it is somewhat peculiar that if G is a logical truth, then the statement that G is a logical truth does not count as a logical truth, but only as a set-theoretical truth"[5].

Note that here the interior expression $\Diamond_{cat,basket}$ makes a claim about the structure of the cats and baskets in whatever possible scenario is being considered. It doesn't preserve the way these terms apply in the actual world[5].

2.3 Motivating Case: Three Colorability

To illustrate how conditional logical possibility claims promise to help us nominalize physical theories while preserving their explanatory and unificatory power, let me begin with a concrete example of a mathematical explanation of physical facts.

Suppose that a certain map (perhaps one with infinitely many countries) has never actually been three-colored. A good explanation for this fact might be that (in a mathematical sense) the map isn't three colorable. A Platonist might express this idea as follows.

> **Platonistic Non-Three-Colorability**: There is no function (in the sense of a set of ordered pairs) which takes countries on the map to numbers $\{1, 2, 3\}$ in a such a way that adjacent countries are always taken to distinct numbers.

However, we now have an additional nominalist version of the above three-colorability explanation to consider.

> **Modal Non-Three-Colorability**: $\neg\Diamond_{adjacent,country}$ Each country is either yellow, green or blue and no two adjacent countries are the same color.

And the above modal explanation can seem to be at least as good, indeed better than the nominalist explanation.

[5]See appendix A for a more technical details.

In particular, one might argue that the Platonistic non-three-colorability claim can only intuitively explain the fact that a physical map is not three-colored because we assume a certain relationship between set existence facts and the modal facts referenced above. Specifically, we assume that there are functions corresponding to *all possible ways* of pairing countries with one of the numbers 1, 2 or 3, and hence all possible ways of choosing how to color these countries. If we suspend judgment on this claim, inference from the non-existence of a certain kind of *set* to the claim that the map isn't actually three-colored begins to look unjustified.

Thus, one might argue that the real explanatory work here is being done by the modal principle; claims about what mathematical objects (e.g. sets coding three coloring functions) exist don't really add anything[6]. One might also claim it as an advantage that the modal nature of the nominalist paraphrase matches ordinary language better than Platonistic paraphrases do. We tend to express thoughts like the non-three-coloring explanation above modally, by talking about maps being three color*able*, rather than ontologically, by talking about maps *having three colorings*.

In any case, I hope considering the above toy explanation provides a motivating example of how logical tools used in potentialist set theory (the conditional logical possibility operator) can help us nominalize Platonist scientific theories in a way that preserves (or improves) their explanatory and unifying power.

3 Nominalist Paraphrase

Now let's turn to the task of providing a general nominalistic paraphrase strategy — which preserves explanatory and unificatory virtues as above. In this section, I'll review a basic modal if-thenist nominalistic paraphrase strategy T,

[6]See [4] for more argument on this point.

which transforms every Platonist sentence ϕ (satisfying a certain definable supervenience condition) into a nominalistically acceptable sentence $T(\phi)$ which Platonists must regard as capturing the non-mathematical content of ϕ (by being true at exactly the same metaphysically possible worlds as ϕ).

My basic strategy follows a familiar modal twist on if-thenism, developed by Putnam and Hellman[18, 22] among others. First, we come up with some axioms completely pinning down the mathematical and quasi-mathematical structures the Platonist (but not the nominalist) believes in. For example, in the case of the natural numbers, these axioms might be a version of the second-order Peano Axioms characterizing the natural number structure (written using the conditional possibility operator[7]). Then, given a sentence ϕ in a scientific theory (e.g., Newton's law of gravitation) we can nominalistically formalize scientists' apparent assertion of ϕ as saying that if there were objects satisfying these axioms then ϕ would be true.[8]

More generally the strategy will be to have $T(\phi)$ assert that ϕ would be true if we supplemented the non-mathematical world with the mathematical (and applied mathematical) objects which the Platonist assumed when asserting ϕ.

To apply this strategy, we need a sentence/axiom D that specifies what relevant mathematical objects (and relations involving them) the Platonist takes there to be in terms of facts about how some nominalistic relations (i.e., relations whose extension the Platonist and nominalist agree on) apply, with the following properties.

- The Platonist takes D to be a metaphysically necessary truth,

[7]See Appendix B for a demonstration of how to replace second-order quantification with the conditional logical possibility operator.

[8]In cases where we have a categorical description of the relevant structure (i.e., any two structures satisfying the description would have to be isomorphic to each other), this gives bivalent truth conditions for all pure mathematical statements. Note that when it's necessary to use second-order quantification to pin down a categorical conception of the relevant structure, we can do this purely in the language of conditional logical possibility as shown in [4].

- D uniquely pins down how all the Platonistic relations (i.e., relations whose extensions the Platonist and nominalist disagree on) in $\vec{P}$ are supposed to apply at each metaphysically possible world – given the facts about some finite list of nominalistic relations $N_1 \ldots N_m$ at that world.

I will call such a sentence D a definable supervenience sentence. And I will say that the application of some Platonistic vocabulary $\vec{P}$ definably supervenes on the application of some nominalistic vocabulary $\vec{N}$ when we can write such a sentence.

So, for example, a definable supervenience sentence for a the Platonistic vocabulary in a scientific theory involving natural numbers will include a categorical description of the natural numbers. And if we want to translate Platonist claims about a (supposed) layer of sets of goats, this definable supervenience claim will typically involve statements will imply that sets (of goats) are extensional and some version of the idea that there's a set of goats corresponding to 'all possible ways of choosing' some of the goats[9].

When we have such a definable supervenience sentence D we can nominalistically translate every sentence ϕ which only employs relations in $\vec{P}, \vec{N}$ (and has all quantifiers restricted to objects related by one of the relations in $\vec{P}, \vec{N}$[10]). For the truth value of all such sentences ϕ will be completely determined by the structure of objects satisfying the Platonistic and nominalistic relations $\vec{P}, \vec{N}$. And one can use the relevant definable supervenience description D to precisely pin down the Platonic structure (at each possible world) in terms of the intended relationship between Platonistic objects and relations and nominalistically acceptable ones. [11]

[9]This concept turns out to be easy to express in the language of conditional logical possibility, as shown in appendix B

[10]More formally, those objects which take part in some tuple satisfying one of these relations.

[11]Note that, as one can categorically specify standard mathematical structures using conditional logical possibility such structures automatically satisfy the definable supervenience condition. See appendix B and [4].

In particular, we can nominalistically paraphrase such a sentence ϕ as follows.

$T(\phi) = \Box_{\vec{N}}(D \to \phi)$

Intuitively, this says that it's logically necessary, given the structure of objects satisfying the nominalistic relations $\vec{N}$, that *if* there were (objects with the intended structure of the) relevant mathematical objects then ϕ would be true[12]

So, for example, consider the statement

GOATS 'There are some goats who admire only each other[13].'

Applying our nominalistic paraphrase strategy will give a sentence T(GOATS) with the following form.

$\Box_{goat,admire}$[There are (objects with the intended structure of) the sets of goats $\to$ There is a set of goats x, such that the goats in x admire only each other.]

This nominalistic paraphrase strategy is good in the following sense. From a nominalist point of view, $T(\phi)$ captures all the non-mathematical content that the Platonist *intended* to express by ϕ. Where it is defined, $T(\phi)$ is true at exactly those metaphysically possible worlds where the Platonist thinks ϕ is true. To put this point another way, if we suppose the Platonist assumptions articulated in the relevant definable supervenience D are metaphysically necessary truths (as the Platonist believes), then it will be metaphysically necessary that ϕ is true if and only if $T(\phi)$. [14]

[12]Note that the Platonist must believe it is always logically possible to supplement the non-mathematical objects at each possible world with additional objects so that D is satisfied, for the Platonist thinks that D is a metaphysically necessary truth.

[13]Here I mean the version of this which a Platonist might express by saying: there's a collection/set of goats which only admire other goats in that collection.

[14]Some nominalists might worry about the above translations' use of mathematical vocabulary like 'set' and 'element' inside the $\Diamond/\Box$ of logical possibility/necessity. For, as stated, my paraphrases make claims about how it would be logically (not to say metaphysically!) possible for there to objects like sets with ur-elements. Nominalists who think 'set' is a meaningful

This strategy has some advantages over other nominalization strategies proposed by Field and Rizza. For example, it always produces finitely stateable theories where it applies, unlike Field's proposal in [10] (which sometimes pairs a Platonist theory with an infinite class of nominalistic statements)[15].

It also has no problem applying to Platonist theories that quantify over arbitrarily large mathematical structures (provided we have a suitable description of them). For, it is logically possible that existing physical structures exist alongside arbitrarily large mathematical structures. This provides an advantage over nominalization strategies like Rizza's [25] which require us to find a copy of whatever mathematical structures the Platonist theory to be paraphrased quantifies over in the physical world. Such limitation on size arguably prevents Rizza's proposal from capturing the unifying explanatory power of Platonist mathematical explanations that appeal to very large mathematical structures.

But how widely can the above paraphrase strategy be applied? For example, can we use it to nominalize all the mathematical explanations for scientific facts that have been used to make explanatory indispensability arguments? If we look at the nice list of such explanations provided by [21], the following picture emerges. The basic modal if-thenist paraphrase strategy stated so far can be immediately applied to about half the cases Lyon mentions. For example, it can be used to nominalistically explain the fact that no walk ever crosses each Köningsburg bridge exactly once – and the same goes for every constellation of more than two islands each of which sports an odd number of bridges.

predicate which just happens to have a necessarily empty extension, this is fine. However, nominalists who aren't fine with this should note that we can entirely banish terms like 'set' and element from the above paraphrases, using any other first-order predicates and relations that don't occur in $\vec{N}$ instead. For example, we could uniformly replace 'set' and 'element' in the translation above with,'angel' and '...is transubstantiated into...' in our $T(\phi)$. This strategy is reminiscent of Putnam's strategy for stating potentialist set theory in [22].

[15]However certain disadvantages may also be admitted. Most obviously, accepting the conditional logical possibility operator is controversial (though, recall, Field himself advocates accepting a primitive logical possibility operator and uses it in his argument for conservatism). Also, the kind of paraphrases of physical magnitude statements provided will not be as attractively 'intrinsic' in the way Field wants.

However, it's not clear that this basic paraphrase strategy can be used to nominalize the other half of the explanations on Lyons' list: the mathematical explanations of physical facts involving distance and other physical magnitudes (for example, Lyon lists an explanation for the hexagonal shape of honeycombs which appeals to the fact that this shape optimizes the ratio of area to perimeter). And as we will see in the next section, it's unclear whether/how this basic paraphrase strategy can be applied to physical magnitude statements (i.e., ones involving relations like 'is n meters long' or 'is m times longer than').

4 Physical Magnitude Statements

In this section I will review some reasons (arising from Putnam's counting argument in [23] and the following literature) why one might fear that no definable supervenience condition can be produced for Platonist theories involving physical magnitudes like mass, charge and length — and that no adequate nominalistic paraphrase of these sentences is possible.

Puntam's counting argument in [23] notes that when formalizing a theory like Newton's law of gravity, the Platonist can appeal to notions like a mass relation which relates physical objects to their mass in grams, or a mass ratio relation which relates pairs of objects to a number that's the ratio between their masses. Using these Platonistic relations (relations to mathematical objects) they can distinguish –and write theories that imply different consequences given – infinitely many different possibilities (w.r.t. the length ratios), in a universe containing only two physical objects. In contrast, any nominalist paraphrase language (that only uses finitely many relations, which only relate physical objects), can only distinguish finitely many distinct possibilities for a world which contains only two physical objects. Accordingly, it seems that there couldn't possibly be any nominalistically acceptable theory which captures the full range of implications about objects standing in various different distance/length ratios which Platonist theories can distinguish.

Field in [10] responded by noting that measurement theoretic uniqueness theorems suggests a solution to this problem – at least as regards the specific notion of length (if we are willing to be substantivalists about space). Given some assumptions, which I'll call the claim that space is richly instantiated[16], we can uniquely pick out the Platonist's intended length-in-meters relation function (from among all other functions from objects to real numbers) by saying it assigns length 1 to some canonical path and assigns lengths in a way that respects the following nominalistic relations:

- $\leq_L$ 'path p_1 is at least as long as path p_2'
- $\oplus_L$ 'the combined lengths of path p_1 and p_2 together are equal to the length of path p_3'[17].

Thus, we have a formula ψ which picks out the Platonist's length-in-meters function at all possible words where length is richly instantiated. So, at all such possible worlds, a Platonist sentence $\phi(l)$ (in the language of set theory with ur-elements, with l being a name for this length function) will be true if and only iff the corresponding nominalist sentence $T(\phi)$ (below) is true.

$T(\phi)$ 'Necessarily if there are objects satisfying our description of the

[16]Specifically, we can prove the uniqueness claim above holds whenever the following three principles (which all happen to be stateable in the language of set theory with ur-elements) are satisfied. My presentation follows [26].

Closure Under Multiples: Given a path x, there are paths y with lengths equal to any finite multiple of the length of x.

Archimedian Assumption: No path is infinite in length with respect to another, i.e., if $x \leq_L y$ then some finite multiple of x is longer than y (i.e. there's a path shorter than y, which can be cut up into n segments each of which has the same length as x.

Relational Properties: The relations $\leq_L, \oplus_L$ have the basic properties you would expect from their role as length comparisons.

[17]I will say a function $l(x)$ respects $\leq_L, \oplus_L$ just if for all paths a, b and c $a \leq_L b \iff l(a) \leq l(b)$ and $\oplus_L(a, b, c) \iff l(a) + l(b) = l(c)$.

hierarchy of sets with ur-elements $V_{\omega+\omega}$ then $(\exists f)(\psi(f) \wedge \phi[l/f])$'

Thus one might hope Platonist appeals to length relations can be harmlessly replaced by the strategy above. And maybe (as Field perhaps suggests in [10]) Platonist talk of mass, charge etc. functions could be handled similarly.

4.1 Sparse Magnitudes Problem

However, a crucial difficulty, which I'll call the Sparse Magnitude problem, remains! For, although lengths are plausibly richly instantiated in our world, it's not clear that they're richly instantiated at all metaphysically possible worlds. And other physical magnitudes, like mass and charge, don't even seem to be richly instantiated in the actual world. Indeed, as Eddon puts it [8] (with slight adjustments to the choice of nominalistic primitives I've used above made in brackets):

> It seems possible for there to be a world, w_1, in which a and b are the only massive objects, and a is [three times] as massive as b. It also seems possible for there to be a world, w_2, in which a and b are the only massive objects, and a is [four] times as massive as b. Worlds w_1 and w_2 are exactly alike with respect to their patterns of [how the relations 'less massive than' $o_1 \leq_M o_2$ and $\oplus_M(o_1, o_2, o_3)$ 'combined mass of a + mass of b = mass of c' apply]. And thus they are exactly alike with respect to the constraints these relations place on numerical assignments of mass. ... So it seems we cannot discriminate between the two possibilities we started out with.

These considerations threaten to block the above nominalist paraphrase strategy by showing that length is a special case. They suggest that other physical magnitudes (like mass) can't be pinned down in the same way that length can, and perhaps that the values of physical magnitudes doesn't supervene on facts about how *any* finite list nominalistic relations) apply[18]. Field notes and discusses a version of this problem in [12] the last chapter of[13].

5 A Solution – In a Sense

5.1 Four Place Relation

I'll now argue that we can solve the above sparse magnitudes problem by using two cheap tricks. Specifically, suppose the Platonist worries that object masses or any other property (given by real numbers) can't be captured by any relations between nominalistically acceptable objects.

First, note that if we (temporarily) assume that length is richly instantiated at all metaphysically possible worlds, we can solve the sparse magnitude problem by using length ratios to nominalistically pin down other physical magnitudes.

For example, the nominalist can pick out the mass function by appeal to a four place relation between pairs of objects with masses and pairs of paths:

- $M(p_1, p_2, m_1, m_2)$ which holds iff the ratio of the masses of m_1 to the mass of m_2 is $\leq$ the ratio of the length of path p_1 to the length of the path p_2.

Although such a relation may not be very physically (or metaphysically) natural, it reflects a genuine nominalistically acceptable fact about the world, and suffices for our purposes. By the measurement theory results mentioned above, we can uniquely pin down the length function (up to a choice of unit), at all worlds where length is richly instantiated. We can then uniquely describe the intended mass in grams function m (in terms of its relationship to the length in meters function l) by saying that, for all objects o_1 and o_2 and paths p

[18] Thus a version of Putnam's famous counting argument in [23] threatens to re-arise, even for those nominalists like Field in [10] who avoid the specific concern about lengths he mentions by accepting the existence of spatial points or paths.

$\frac{m(p_1)}{m(p_2)} \leq \frac{l(o_1)}{l(o_2)}$ iff $M(p_1, p_2, o_1, o_2)$ and that it assigns a unit object to 1. For, note that any attempt to assign the wrong mass ratio r' to a pair of objects m_1, m_2 with mass ratio r can be ruled out by considering paths p_1, p_2 whose length ratio falls between that of r and r' and noting that $\mathscr{M}$ fails the above condition for a pair of paths such that $l(p_1)/l(p_2)$ falls between r and r'. And the existence of such a pair of paths is guaranteed by the assumption that length is richly instantiated (which, indeed, implies that length ratios are dense in $\mathbb{R}$).

This, in turn, is enough to allow us to apply the paraphrase strategy discussed above to claims involving a mass function (and the same goes for other physical quantities).

Importantly, even if length isn't *necessarily* richly instantiated, the modal if-thenist paraphrase strategy described above still gives the correct truth-values in those worlds where length is richly instantiated.

5.2 Holism trick

Now what about the above assumption that length is *metaphysically necessarily* richly instantiated? If one accepted substantivalism about space, as I am doing for the purposes of this paper (much as Field does in [10]), there is some attraction to this assumption. However, certain trends in physics raise a worry about this. For, physicists do seem to consider hypotheses on which *space itself* is quantized, so that that length isn't richly instantiated (even from a substantivalist point of view, where spatial points and paths exist and hence can stand in length relations). And we might want to say this kind of epistemic legitimacy (quantized space not being ruled out a priori) suggests we should regard quantized space as a genuine metaphysical possibility. Thus even if we think that space is actually richly instantiated (as it seems to be), we might want to deny that length is metaphysically necessarily richly instantiated.

Happily, however, it turns out that we can adequately nominalize our overall scientific theory (in the sense specified above) even if length isn't metaphysically necessarily richly instantiated. For (if formalized in the natural way for a substantivalist about space) our best current physical theories imply that length is richly instantiated. And it turns out to be particularly easy to nominalize theories which imply that length is richly instantiated.

We have just seen how to produce a partially accurate paraphrase $T(\phi)$, which gets the correct truth-value at worlds where length is richly instantiated, but (for all I've said so far) may get the wrong truth value at other possible worlds. And we can also write a completely correct nominalistic paraphrase of the claim that space is richly instantiated (call this R)[19].

Thus, we can create a nominalistic sentence which (the Platonist must think) has the truth same truth value as ϕ at *all* possible worlds, by simply conjoining these claims.

$T^*(\phi)$: $T(\phi) \wedge R$

At worlds where length is richly instantiated, $T^*(\phi)$ has the correct truth value by our initial point, and R is true at those worlds, so the above conjunction will have the correct truth value. And at worlds where space isn't richly instantiated R is false, hence so is our paraphrase. Thus, in both cases, our paraphrase has the intended truth value.

Accordingly, I claim that a nominalist plausibly *can* address the sparse magnitude problems sufficiently well to answer the classic Quinean indispensability argument.

In fact, it turns out we can improve on this solution a bit, by dropping the ugly expedient of conjoining R and deploying a more uniform paraphrase strategy (i.e., one that's more continuous with the simple strategy for para-

[19]Note that this claim is statable using only set theory with ur-elements and the relations $\leq_L, \oplus_L$, so our basic modal if-thenist strategy suffices to paraphrase it.

phrasing statements about sets of goats presented in §4). For attractive and relatively uncontroversial modal reasoning implies that, (when combined with the bootstrapping trick above) our basic modal if-thenist paraphrase strategy $T(\phi)$ *already* yields an adequate nominalization for all theories ϕ implying space is richly instantiated (i.e., $T(\phi)$ has the same truth value as ϕ at all possible worlds, according to the Platonist). [20]

6 Conclusion

In this paper, I have argued that a mathematical nominalist (who accepts certain notions independently motivated by the literature on potentialist set theory) can plausibly answer both classic Quinean and explanatory indispensability worries raised by scientific use of physical magnitude statements by deploying certain cheap tricks[21].

[20] Thanks to an anonymous reviewer for this point. When T implies that length is richly instantiated, we can show that $P^*(T)$ is already false at all worlds where length is not richly instantiated. For if length is not richly instantiated, then it's logically necessary given the facts about how all nominalistic relations $\vec{N}$ (which include the length ratios relations) apply that length is not richly instantiated. So we have $\Box_{\vec{N}}\neg R$ and $\Box_{\vec{N}}(D \to \neg R)$. By assumption, T implies that length is richly instantiated. So $(T \wedge D) \to R$ is a logical truth, which holds in all logically possible situations. Thus, we can infer $\Box_{\vec{N}}(D \to \neg T)$ (i.e., the claim that $P*(\neg T)$ is true). To get the claim that $P^*(T)$ is false, note that it's logically possible to satisfy the Platonist's assumptions D about how non-mathematical reality and mathematical objects relate, at every metaphysically possible world. So, at the worlds in question, we have $\Diamond_{\vec{N}}D$. And, by the necessity claim above, it follows that $\Diamond_{\vec{N}}(D \wedge \neg T)$. Suppose for contradiction that $P^*(T)$ was true at these worlds, i.e., $\Box_{\vec{N}}(D \to T)$ which is equivalent to $\neg\Diamond_{\vec{N}}(D \wedge \neg T)$. Thus $P^*(T)$ is false at all worlds where length is richly instantiated. See [4] for a more formal treatment, using an axiomatic system for reasoning about conditional logical possibility.

[21] However, I don't think this shows that all is plain sailing for the nominalist. For example, note that the four-place relations that I've invoked are not very metaphysically elegant, and hence are poor candidates for grounding facts about physical magnitudes. Accordingly, something like a grounding indispensability worry may remain ('if there aren't numbers related to objects via a mass ratio-relation, what grounds mass facts?'), even if we can solve classic and explanatory indispensability arguments by logically regimenting our scientific theories involving physical magnitudes in the way I've suggested.

A Set Theoretic Mimicry

Although the conditional logical possibility operator is proposed as a conceptual and metaphysical primitive, we can use the familiar formal background of set theory to *mimic* intended truth conditions for statements in a language containing the logical possibility operator $\Diamond$ alongside usual first order logical vocabulary (where distinct relation symbols R_1 and R_2 always express distinct relations) as follows.

> A formula ψ is true relative to a model $\mathscr{M}$ ($\mathscr{M} \models \psi$) and an assignment ρ which takes the free variables in ψ to elements in the domain of $\mathscr{M}$[22] just if:
>
> - $\psi = R^k_n(x_1 \dots x_k)$ and $\mathscr{M} \models R^k_n(\rho(x_1), \dots, \rho(x_k))$.
> - $\psi = x = y$ and $\rho(x) = \rho(y)$.
> - $\psi = \neg\phi$ and ϕ is not true relative to $\mathscr{M}, \rho$.
> - $\psi = \phi \wedge \psi$ and both ϕ and ψ are true relative to $\mathscr{M}, \rho$.
> - $\psi = \phi \vee \psi$ and either ϕ or ψ are true relative to $\mathscr{M}, \rho$.
> - $\psi = \exists x \phi(x)$ and there is an assignment ρ' which extends ρ by assigning a value to an additional variable v not in ϕ and $\phi[x/v]$ is true relative to $\mathscr{M}, \rho'$[23].
> - $\psi = \Diamond_{R_1 \dots R_n}\phi$ and there is another model $\mathscr{M}'$ which assigns the same tuples to the extensions of $R_1 \dots R_n$ as $\mathscr{M}$ and $\mathscr{M}' \models \phi$.[24]

Note that this means that $\bot$ is not true relative to any model $\mathscr{M}$ and assignment ρ.

[22]Specifically: a partial function ρ from the collection of variables in the language of logical possibility to objects in $\mathscr{M}$, such that the domain of ρ is finite and includes (at least) all free variables in ψ

[23]As usual (?) $\phi[x/v]$ substitutes v for x everywhere where x occurs free in ϕ

[24]As usual, I am taking $\Box$ to abbreviate $\neg\Diamond\neg$

If we ignore the possibility of sentences which demand something coherent but fail to have set models because their truth would require the existence of too many objects, we could then characterize logical possibility as follows:

> **Set Theoretic Approximation**: A sentence in the language of logical possibility is true (on some interpretation of the quantifier and atomic relation symbols of the language of logical possibility) iff it is true relative to a set theoretic model whose domain and extensions for atomic relations captures what objects there are and how these atomic relations actually apply (according to this interpretation) and the empty assignment function ρ.

B Translation Strategy Details

To make my proposed basic modal if-thenist paraphrase strategy T more precise, I'll start by specifying some definitions used to state the definable supervenience condition described above.

B.1 Nominalistic vs. Platonistic Vocabulary

A relation R counts as **nominalistic** vocabulary iff the Platonist and nominalist that it only applies to non-mathematical objects (and about how it applies). So, for example, 'is a cat' and 'is taller than' are nominalistic relations. Platonistic vocabulary is all vocabulary that isn't nominalistic. So for example 'is a number', 'is an element of', 'is a set of goats', 'is a function from the cats to numbers' and '...has more than...fleas' are all Platonistic vocabulary.

B.2 Categoricity Over

Next, we want to express the idea that some description D 'specifies, for each possible world w, exactly what mathematical objects the Platonist thinks exist at w (and how all relevant Platonistic vocabulary applies)', so that D can be a suitable antecedent for our if-thenist translation.

First, I will expand the notion of categoricity (all models of some theory are isomorphic) to a notion of **categoricity for** some list of relations **over** some other list of relations. I will say that a description $D(N_1, \ldots, N_m, P_1, \ldots, P_n)$ is categorical for the relations $P_1, \ldots, P_n$ over the relations $N_1, \ldots, N_m$ when (for every logically possible way the relations $N_1, \ldots, N_m$ could apply), requiring that D suffices to pin down a unique overall structure of objects satisfying relations in $P_1, \ldots, P_n, N_1, \ldots, N_m$. So stipulating that D uniquely determines (given the facts about how some relations nominalistic relations $N_1, \ldots, N_m$), how the objects related by these relations could be supplemented by additional objects satisfying Platonistic relations $P_1, \ldots, P_n$[25].

For example, the following sentence D: SETS OF GOATS categorically describes how the Platonistic relations 'is a set-of-goats' and '...is an element of set-of-goats...' apply over the nominalistic relations 'is a goat'.

D: SETS OF GOATS

- The sets of goats are extensional[26].
- It's logically necessary, given the facts about how 'is a goat' 'is a set-of-goats' and '...is an element of set-of-goats...' are supposed to apply at any possible world, that if some goats are

[25] So, for example, if the sets of people, along with set membership, $(S_{\text{people}}, \in_{\text{people}})$ is categorical over the people P it's not just true that the number of sets of people is totally determined by what people exist but also facts such as whether or not any set of people is a person must also be determined. This claim can be nicely articulated in the language of logical possibility, as shown in [4].

[26] That is, sets of goats a and b are identical just if they have exactly the same members.

happy then there's a set of goats whose elements are exactly the happy goats.

- No goat is a set-of-goats.

- If x is an element of set-of-goats y, then x is a goat and y is a set-of-goats.

B.3 Definable Supervenience

Now we can state the definable supervenience condition as follows.

A list of relations $\vec{P}$ **definably supervenes** (via a sentence D) on a finite list of nominalistic relations $\vec{N}$ iff

- There's a sentence D (a 'Supervenience Description' that intuitively explains how the relevant Platonistic facts supervene on nominalistic facts) in the language of logical possibility[27] which satisfies the following conditions

  - D is formed using only relations in $\vec{P}, \vec{N}$ and all quantifiers in D are restricted to objects that satisfy at least one relation in this collection[28]

  - The Platonist being translated takes D to express a metaphysically necessary truth.

  - $\Box\Diamond_{\vec{N}}D$, i.e., the Platonist isn't supposing the existence of incoherent objects and indeed it's logically necessary that the $\vec{N}$ structure can be supplemented with Platonistic structure in the way that D requires. $\vec{P}, \vec{N}$

[27]So D employs only the FOL logical connectives and the conditional logical possibility operator as logical vocabulary, and does not quantify in to the $\Diamond$ of logical possibility[4]

[28]The latter assumption ensures that D 'only talks about' the structure of objects satisfying relations in P and N.

– D is is categorical for the relations $P_1, \ldots, P_n$ over the relations $N_1, \ldots, N_m$

For example, in the case above, note that the Platonist takes $D_{\text{Sets of Goats}}$ to be a metaphysically necessary truth. And $D_{SetsofGoats}$ specifies exactly what sets of goats there are at each metaphysically possible world w (and the elementhood relation on these sets), given the facts about what goats there are at each world. Also, it's logically necessary that, however the goats are configured, they can be supplemented with sets as required by $D_{\text{Sets of Goats}}$.

Surprisingly, many collections of Platonistic sentences involving pure mathematical structures (of reals, complex numbers etc.) and applied mathematical objects (of classes of physical objects, functions from physical objects to pure mathematical objects) straightforwardly satisfy this definable supervenience condition.

For example, we can create a definable supervenience description D for translating sentences that talk about both numbers and the objects satisfying some nominalistic relations $\mathscr{N}$, by conjoining claims that the natural numbers are distinct from all the objects related by these nominalistic relations with a sentence $PA_{\Diamond}$ that categorically describes the natural numbers over N.

And we can write a sentence $PA_{\Diamond}$ (using the conditional logical possibility operator) that categorically describes the intended structure of the natural numbers $\mathbb{N}, \mathbb{S}$ over any list of relations, including the empty list of relations[29] as follows. First not that we can categorically describe the natural numbers via the second order Peano Axioms, a combination of all the first order Peano Axioms except for instances of the induction schema conjoined with the following second order statement of induction.

[29]Unsurprisingly we don't need to appeal to facts about how any nominalistic relations happen to apply in order to pin down the intended structure of these pure mathematical objects.

$$(\forall X)\,[(X(0) \wedge (\forall n)\,(X(n) \rightarrow X(n+1))) \rightarrow (\forall n)(X(n))]$$

We can reformulate this claim using conditional logical possibility as follows[30].

- '$\Box_{\mathbb{N},S}$ If 0 is happy and the successor of every happy number is happy then every number is happy.

In other words: it is logically necessary, given how $\mathbb{N}$ and S apply, then if 0 is happy and the successor of every happy number is happy then every number is happy.'

Thus, we can write a sentence $PA_{\Diamond}$, (purely in terms of first order logic plus the conditional logical possibility operator) which categorically describes the natural numbers. Just use the fact above to replace the second-order induction axiom in second order Peano Arithmetic with a version stated in terms of conditional logical possibility. Recall that the Second Order Peano Axioms are the familiar first order Peano Axioms for number theory, but with the induction schema replaced by a single induction axiom using second order quantification. In [4] I argue that we can similarly rewrite other second-order conceptions of pure mathematical structures.

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