Chapter 8

Useful Corollaries to Axioms

8.1 Diamond Simplification Lemmas

Lemma 8.1.1. ***Basic Diamond Simplification*** $\vdash \Diamond_{\mathcal{L}}(\Diamond_{\mathcal{L},R_1}(\phi)) \rightarrow \Diamond_{\mathcal{L}}\phi$

Proof. Suppose $\Diamond_{\mathcal{L}}(\Diamond_{\mathcal{L},R1}(\phi))$. First we enter the outer $\Diamond_{\mathcal{L}}$ context, beginning an In$\Diamond$ argument. Since we have $\Diamond_{\mathcal{L},R_1}(\phi)$ in this context, we can apply ignoring to deduce $\Diamond_{\mathcal{L}}(\phi)$. Thus, leaving the above special context we have $\Diamond_{\mathcal{L}}(\Diamond_{\mathcal{L}}(\phi))$. Now the inside statement is content-restricted to $\mathcal{L}$, so by $\Diamond$E we can infer from its logical possibility (given the facts about $\mathcal{L}$ to its actuality). This gives us $\Diamond_{\mathcal{L}}\phi$, as desired.

1		$\Diamond_{\mathcal{L}}(\Diamond_{\mathcal{L},R_1}(\phi))$	[1]
2	$\Diamond$	$\Diamond_{\mathcal{L},R_1}(\phi)$ $[\mathcal{L}]$	1, In$\Diamond$I [1]
3		$\Diamond_{\mathcal{L}}(\phi)$	2, Ign I [1]
4		$\Diamond_{\mathcal{L}}(\Diamond_{\mathcal{L}}\phi)$	1,2-3 In$\Diamond$E [1]
5		$\Diamond_{\mathcal{L}}\phi$	4 $\Diamond$ E [1]

□

Lemma 8.1.2. ***Diamond Collapsing:*** *If ϕ_2 and θ are content restricted to $\mathcal{L}_1, \mathcal{L}_2$ and ϕ_1 is content restricted to $\mathcal{L}_0, \mathcal{L}_1$, then we have*

$$\vdash \Diamond_{\mathcal{L}_0}(\phi_1 \wedge \Diamond_{\mathcal{L}_1}(\phi_2 \wedge \theta)) \leftrightarrow \Diamond_{\mathcal{L}_0}(\phi_1 \wedge \phi_2 \wedge \theta)$$

Proof. LTR direction:

Assume $\Diamond_{\mathcal{L}_0}(\phi_1 \wedge \Diamond_{\mathcal{L}_1}(\phi_2 \wedge \theta))$. Enter the $\Diamond_{\mathcal{L}_0}$ context. We have $\Diamond_{\mathcal{L}_1}(\phi_2 \wedge \theta)$. Because $\phi_2 \wedge \theta$ is content restricted to $\mathcal{L}_1, \mathcal{L}_2$, we can use ignoring to turn this into $\Diamond_{\mathcal{L}_0,\mathcal{L}_1}(\phi_2 \wedge \theta)$. Now enter this $\Diamond_{\mathcal{L}_0,\mathcal{L}_1}$ context. We can import ϕ_1 because it is content restricted to $\mathcal{L}_0, \mathcal{L}_1$. Thus we can deduce $\phi_1 \wedge \phi_2 \wedge \theta$.

Leaving this $\Diamond$ context (completing our inner $\Diamond$ argument), we have $\Diamond_{\mathcal{L}_0,\mathcal{L}_1}\phi_1 \wedge \phi_2 \wedge \theta$. Hence we can deduce $\Diamond_{\mathcal{L}_0}\phi_1 \wedge \phi_2 \wedge \theta$ by Ign. Noting that this latter claim is content-restricted to $\mathcal{L}_0$ lets us complete our larger $\Diamond$E argument by pulling the fact that $\Diamond_{\mathcal{L}_0}(\phi_1 \wedge \phi_2 \wedge \theta)$ outside of the outer $\Diamond_{\mathcal{L}_0}$ context.

RTL direction:

Conversely, suppose that $\Diamond_{\mathcal{L}_0}(\phi_1 \wedge \phi_2 \wedge \theta)$. Enter this $\Diamond_{\mathcal{L}_0}$ for Inn$\Diamond$. By $\Diamond$I we can infer from $\phi_2 \wedge \theta$ to $\Diamond_{\mathcal{L}_0}(\phi_2 \wedge \theta)$. Thus we have $\phi_1 \wedge \Diamond_{\mathcal{L}_0}(\phi_2 \wedge \theta)$ and completing our In$\Diamond$ gives $\Diamond_{\mathcal{L}_0}(\phi_1 \wedge \Diamond_{\mathcal{L}_1}(\phi_2 \wedge \theta))$ as desired.

$\Box$

8.2 $\Box$ Ignoring

($\Box$ Ign) $\Box$ Ignoring. If θ is content-restricted to $\mathcal{L}, R_1, \ldots R_n$ and $S_1 \ldots S_m$ are relations not among $\mathcal{L}, R_1, \ldots R_n$ then $\vdash \Box_{\mathcal{L}, S_1 \ldots S_m}\theta \leftrightarrow \Box_{\mathcal{L}}\theta$.

1	$\Box_{\mathcal{L}}\theta$	[1]
2	$\neg\Diamond_{\mathcal{L}}\neg\theta$	[1]
3	$\Diamond_{\mathcal{L}}\neg\theta \leftrightarrow \Diamond_{\mathcal{L},S_1\ldots S_m}\neg\theta$	Ign$\Diamond$
4	$\neg\Diamond_{\mathcal{L},S_1\ldots S_m}\neg\theta$	2,3 FOL [1]
5	$\Box_{\mathcal{L},S_1\ldots S_m}\theta$	[1]
6	$\Box_{\mathcal{L}}\theta \rightarrow \Box_{\mathcal{L},S_1\ldots S_m}\theta$	5 $\rightarrow$I
7	$\Box_{\mathcal{L},S_1\ldots S_m}\theta$	[7]
8	$\neg\Diamond_{\mathcal{L},S_1\ldots S_m}\neg\theta$	[7]
9	$\neg\Diamond_{\mathcal{L}}\neg\theta$	3,8 FOL [7]
10	$\Box_{\mathcal{L},S_1\ldots S_m}\theta \rightarrow \Box_{\mathcal{L}}\theta$	9 $\rightarrow$I
11	$\Box_{\mathcal{L}}\theta \leftrightarrow \Box_{\mathcal{L},S_1\ldots S_m}\theta$	6,10 FOL

8.3 $\Box$ Collapsing Lemma

If ϕ_2 and θ are content restricted to $\mathcal{L}_1, \mathcal{L}_2$ and ϕ_1 is content restricted to $\mathcal{L}_0, \mathcal{L}_1$, then we have

$\vdash \Box_{\mathcal{L}_0}(\phi_1 \rightarrow \Box_{\mathcal{L}_1}(\phi_2 \rightarrow \theta)) \leftrightarrow \Box_{\mathcal{L}_0}(\phi_1 \wedge \phi_2 \rightarrow \theta)$

LTR direction:

Assume $\Box_{\mathcal{L}_0}(\phi_1 \rightarrow \Box_{\mathcal{L}_1}(\phi_2 \rightarrow \theta))$.

To prove that $\Box_{\mathcal{L}_0}(\phi_1 \wedge \phi_2 \to \theta)$, we consider an arbitrary scenario in which $\phi_1 \wedge \phi_2$ (and the $\mathcal{L}_0$ facts are held fixed).[1] Our initial assumption that $\Box_{\mathcal{L}_0}(\phi_1 \to \Box_{\mathcal{L}_1}(\phi_2 \to \theta))$ is content restricted to $\mathcal{L}_0$, so it must remain true in this scenario. But what is necessary must be actual, so by $\Box$E we can infer $\phi_1 \to \Box_{\mathcal{L}_1}(\phi_2 \to \theta)$. Combining this with our knowledge that ϕ_1 (in the scenario now under consideration), gives $\Box_{\mathcal{L}_1}(\phi_2 \to \theta)$. Again, what is necessary is actual, so we have $(\phi_2 \to \theta)$, and hence we can derive that θ.

Now, discharging our assumption for $\to$I gives us $\phi_1 \wedge \phi_2 \to \theta$. And since we considered an arbitrary situation in which the facts about $\mathcal{L}_0$ were held fixed, we have $\Box_{\mathcal{L}_0}(\phi_1 \wedge \phi_2 \to \theta)$ as desired, by $\Box$I.

[1] That is to say, we enter a $\Box$I context which holds fixed $\mathcal{L}_0$ and assume for $\to$I that $\phi_1 \wedge \phi_2$.

1	$\Box_{\mathcal{L}_0}(\phi_1 \to \Box_{\mathcal{L}_1}(\phi_2 \to \theta))$	[1]
2	$\Box$ $[\mathcal{L}_0]$	
3	$\phi_1 \wedge \phi_2$	[3]
4	$\Box_{\mathcal{L}_0}(\phi_1 \to \Box_{\mathcal{L}_1}(\phi_2 \to \theta))$	1, import [1]
5	$\phi_1 \to \Box_{\mathcal{L}_1}(\phi_2 \to \theta)$	4 $\Box$E [1]
6	$\Box_{\mathcal{L}_1}(\phi_2 \to \theta)$	3,5 FOL [1,3]
7	$\phi_2 \to \theta$	6 $\Box$E [1,3]
8	θ	3,7 FOL [1,3]
9	$\phi_1 \wedge \phi_2 \to \theta$	3,8 $\to$I [1]
10	$\Box_{\mathcal{L}}(\phi_1 \wedge \phi_2 \to \theta)$	2-5 $\Box$I [1]

RTL direction:

Conversely, assume $\Box_{\mathcal{L}_0}(\phi_1 \wedge \phi_2 \to \theta)$

To prove that $\Box_{\mathcal{L}_0}(\phi_1 \to \Box_{\mathcal{L}_1}(\phi_2 \to \theta))$, we consider an arbitrary scenario in which ϕ_1 and the $\mathcal{L}_0$ facts are held fixed. Our initial assumption above is content-restricted to $\mathcal{L}_0$, so it must remain true in this scenario.

Then we consider a further arbitrary scenario in which ϕ_2 (while the application of $\mathcal{L}_0, \mathcal{L}_1$ in the scenario above is held fixed). Since ϕ_1 held true in the previous scenario, and it is content restricted to $\mathcal{L}_0, \mathcal{L}_1$ it must remain true in this second scenario. Thus we have $\phi_1 \wedge \phi_2$. Similarly, since our initial assumption that $\Box_{\mathcal{L}_0}(\phi_1 \wedge \phi_2 \rightarrow \theta)$ was true in the previous scenario and it is content-restricted to $\mathcal{L}_0, \mathcal{L}_1$, it must also remain true in the scenario currently under consideration. And since what is necessary is actual, we can derive $\phi_1 \wedge \phi_2 \rightarrow \theta$. Putting this together with $\phi_1 \wedge \phi_2$ gives us that θ is true in the scenario under consideration.

Now in the previous paragraph, we have shown that an arbitrary scenario in which the $\mathcal{L}_0, \mathcal{L}_1$ facts from our first scenario are preserved and ϕ_2 holds true must also be one in which θ. Thus we know that our first scenario was one in in which $\Box_{\mathcal{L}_0,\mathcal{L}_1}(\phi_2 \rightarrow \theta)$, by conditional proof and then $\Box$I. And since $\phi_2 \rightarrow \theta$ is content-restricted to $\mathcal{L}_1$, we can use (the $\Box$ version of) ignoring deduce that $\Box_{\mathcal{L}_1}(\phi_2 \rightarrow \theta)$.

Thus we have shown that an arbitrary scenario in which ϕ_1 is true and the $\mathcal{L}_0$ facts are held fixed must be one in which $\Box_{\mathcal{L}_1}(\phi_2 \rightarrow \theta)$. From this it follows by $\Box$I and conditional proof that $\Box_{\mathcal{L}_0}(\phi_1 \rightarrow \Box_{\mathcal{L}_1}(\phi_2 \rightarrow \theta))$ as desired.

1		$\Box_{\mathcal{L}_0}(\phi_1 \wedge \phi_2 \to \theta)$	assump. [1]
2	$\Box$	$[\mathcal{L}_0]$	
3		$\Box_{\mathcal{L}_0}(\phi_1 \wedge \phi_2 \to \theta)$	1 import [1]
4		ϕ_1	assump. [3]
5	$\Box$	$[\mathcal{L}_0, \mathcal{L}_1]$	
6		ϕ_2	assump. [6]
7		ϕ_1	4 import [3]
8		$\phi_1 \wedge \psi$	6, 7 FOL [3,6]
9		$\Box_{\mathcal{L}_0}(\phi_1 \wedge \phi_2 \to \theta)$	3 import [1]
10		$\phi_1 \wedge \phi_2 \to \theta$	9 $\Box$E [1]
11		θ	8,10 FOL [1,3,6]
12		$\phi_2 \to \theta$	6,11 $\to$ I [1,3]
13		$\Box_{\mathcal{L}_0,\mathcal{L}_1}(\phi_2 \to \theta)$	5-12 $\Box$I [1,3]
14		$\Box_{\mathcal{L}_1}(\phi_2 \to \theta)$	13 $\Box$Ign [1,3]
15		$\phi_1 \to \Box_{\mathcal{L}_1}(\phi_2 \to \theta)$	3,14 $\to$I [1]
16		$\Box_{\mathcal{L}_0}(\phi_1 \to \Box_{\mathcal{L}_1}(\phi_2 \to \theta))$	2-15 $\Box$I [1]

Putting these two arguments together in the obvious first order logical way gives us $\Box_{\mathcal{L}_0}(\phi_1 \to \Box_{\mathcal{L}_1}(\phi_2 \to \theta)) \leftrightarrow \Box_{\mathcal{L}_0}(\phi_1 \wedge \phi_2 \to \theta)$.

8.4 Box Relabeling

Lemma 8.4.1. ***Box Relabling*** *If $R_1 \dots R_n$ are relations that occur in θ but not in $\mathcal{L}$, and $R'_1 \dots R'_n$ are relations with the same arities (i.e., the arity of R_i and R'_i are the same) that don't occur in $\mathcal{L}$ or θ, then $\Gamma \vdash \Box_{\mathcal{L}}\theta \leftrightarrow \Box_{\mathcal{L}}\theta[R_1/R'_1 \dots R_n/R'_n]$.*

Proof. We can prove this straighforwardly from Relabling and the fact that $\Box$ abbreviates $\neg \Diamond \neg$

1	$\Diamond_{\mathcal{L}}\neg\theta \leftrightarrow \Diamond_{\mathcal{L}}\neg\theta[R_1/R'_1 \dots R_n/R'_n]$	ReL
2	$\neg \Diamond_{\mathcal{L}} \neg\theta \leftrightarrow \neg \Diamond_{\mathcal{L}} \neg\theta[R_1/R'_1 \dots R_n/R'_n]$	1, Fol
3	$\Box_{\mathcal{L}}\theta \leftrightarrow \Box_{\mathcal{L}}\theta[R_1/R'_1 \dots R_n/R'_n]$	by def of box

□

8.5 Multiple Definitions Lemma

Lemma 8.5.1. ***Multiple Definition Lemma****: Often we will want to make a chain of explicit definitions – to using Simple Comprehension or Modal Comprehension or Choice to specify the application of a series of relations $R_1 ... R_n$ in turn. Thus we have*

- ϕ
- $\Diamond_{\mathcal{L}}\psi_1$, *where* ψ_1 *specifies a way that* R_1 *could apply in terms of* $\mathcal{L}$ *(so* ψ_1 *content-restricted to* $\mathcal{L}, R_1$*),*
- *inside this* $\Diamond$ *context* $\Diamond_{\mathcal{L},R_1}\psi_2$ *where* ψ_2 *specifies a way that* R_2 *could apply in terms of* $\mathcal{L}, R_1$ *(so* ψ_2 *content-restricted to* $\mathcal{L}, R_1, R_2$*)*
- *etc.*

And we can hence conclude that $\Diamond_{\mathcal{L}}(\phi \wedge \psi_1 \wedge \Diamond_{\mathcal{L},R_1}(\psi_2 \wedge \Diamond_{\mathcal{L},R_1,R_2}(\psi_3 \wedge ...)))$.

In such cases we can infer the logical possibility of a single scenario $\Diamond_{\mathcal{L}}(\phi \wedge \psi_1 \wedge ...\psi_n)$

Proof. The desired conclusion follows immediately by repeated application of FOL to suitable instances of the $\Diamond$-collapsing lemma above. □

8.6 Simplified Choice

Simple Choice $\vdash (\exists x)P(x) \rightarrow \Diamond_P(\exists x(P(x) \wedge P'(x) \wedge (\forall y)[P'(y) \rightarrow x = y])$

Suppose for →I, that $(\exists x)P(x)$.

We can use the Possible Powerset axiom schema to get the possibility that $class()$and $\in$ behave like a layer of classes over the objects satisfying P and there is an object which behaves like the $\varnothing$ alongside the objects satisfying P. Enter this $\Diamond_P$-context and use Simple Comprehension to set $(\forall x)(F(x) \leftrightarrow x = \varnothing)$[2] and then (entering this $\Diamond_{P,class,\in}$-context), the possibility that R

[2]Here and in the rest of the proof I will use claims of the form $\phi(\varnothing)$ to abbreviate claims that everything which behaves like the empty set satisfies ϕ i,e. claims of the form $(\exists x)[class(x) \wedge \forall y \neg y \in x \wedge \phi(x)]$.

relates $\varnothing$ to each object satisfying P [i.e., $(\forall x)(\forall y)R(x,y) \leftrightarrow x = \varnothing \wedge P(y)$]. Enter that $\Diamond_{P,class,\epsilon,F}$-context.

Now apply Choice to get the $\Diamond_{F,R}$ of an R' which takes the single object in its domain ($\varnothing$) to a single object. By Ignoring (and the fact that the formula $\forall x \forall y(R'(x,y) \to R(x,y)) \wedge [\forall x F(x) \to \exists! y R'(x,y)$ is content restricted to F,R) we can conclude that the above scenario is also $\Diamond_{P,class,\epsilon,F,R}$. Enter the latter $\Diamond$. By simple comprehension we can have $\Diamond_{P,class,\epsilon,R,F,R'}$ P' applies to the single object which R' relates $\varnothing$ to.

Enter this final $\Diamond$ context. Because our biconditionals characterizing R, F and R' are suitably content-restricted, we can import them through all the $\Diamond$s for use in the current $\Diamond_{P,R,F,R'}$ context. Thus we can deduce that $(\exists x)(P(x) \wedge P'(x) \wedge (\forall y)[P'(y) \to x = y])$ is true in this $\Diamond_{P,class,\epsilon,R,FR'}$ context.

Leaving this context, we can conclude that $\Diamond_P(\exists x)(P(x) \wedge P'(x) \wedge (\forall y)[P'(y) \to x = y])$ by $\Diamond$E. Now this claim is content restricted to P, so we can pull it out of all the various $\Diamond$ contexts (each of which holds fixed the application of P) one by one.

Thus, we can conclude $\vdash (\exists x)P(x) \to \Diamond_P(\exists x(P(x) \wedge P'(x) \wedge (\forall y)[P'(y) \to x = y])$, as desired.

Simple Choice for N-tuples $\vdash (\exists \vec{x})R(\vec{x}) \to \Diamond_R(\exists \vec{x}(R'(\vec{x}) \wedge (\forall \vec{y})[R'(\vec{y}) \to \vec{x} = \vec{y}])$

We can prove all claims of this form by applying the following strategy. First suppose for $\to$I, that $(\exists \vec{x})R(\vec{x})$.

Now apply Possible Powerset a bunch of times (holding fixed R and entering $\Diamond$s after each time) until you have enough layers of sets to have sets corresponding to $\vec{x}$ (as per the usual set theoretic way of associating ordered n-tuples with sets). By simple comprehension, P could apply to exactly those sets coding ntuples $\vec{x}$ such that $R\vec{x}$. Enter this $\Diamond_{R,set1,set2....setn}$ context. By the previous lemma we have $\Diamond_P(\exists x(P(x) \wedge P'(x) \wedge (\forall y)[P'(y) \rightarrow x = y])$. By ignoring we can make this $\Diamond_{P,R,set_1,set_2....set_n}$. Enter the latter $\Diamond$ context. All the facts characterizing the $sets_i$ are suitably content-restricted, so they can be imported. By simple comprehension, it is also logically possible (fixing all the relations mentioned above) that R' applies to exactly single n-tuple $\vec{x}$ coded by the unique set which P' applies to. So, by importing all the previously mentioned facts characterizing R, P, P' and the set_i, and then applying a bunch of first order logic we can derive that $(\exists\vec{x}(R(\vec{x}) \wedge R'(\vec{x}) \wedge (\forall\vec{y})[R'(\vec{y}) \rightarrow \vec{x} = \vec{y}])$.

Finally, we can leave the above $\Diamond$ context and conclude that $\Diamond_R(\exists\vec{x}(R'(\vec{x}) \wedge (\forall\vec{y})[R'(\vec{y}) \rightarrow \vec{x} = \vec{y}])$, by In$\Diamond$. Since this formula is content restricted to R, so we can bring it out of all the $\Diamond$ contexts we have entered (all of which hold fixed R), just as above.

This gives us $\Diamond_R(\exists\vec{x}(R'(\vec{x}) \wedge (\forall\vec{y})[R'(\vec{y}) \rightarrow \vec{x} = \vec{y}])$, and thus the desired conditional.