Chapter 3

The Language of Logical Possibility

Let me begin by introducing the concepts and notation I will use. Speaking generally, potentialist paraphrases of set theory make claims about how it would be logically possible to extend an initial segment of the hierarchy of sets.

Geoffrey Hellman's *Mathematics Without Numbers* [3] influentially formulated a version of potentialism using a logical possibility operator $\Diamond$, together with first and second order quantifiers which are allowed to reach inside the $\Diamond$ (so that we can say things like $\exists x \Diamond \phi$, and $\forall X \forall f \Diamond \psi$). However, it turns out to be possible to simplify this proposal. I will articulate a potentialist explication for set theory using only first order vocabulary and a single, fairly intuitive, notion of relativizable logical possibility (and not allowing quantifying in to the $\Diamond$).

Doing this will allow us to streamline our inference rules and sidestep the controversies about quantifying in discussed above.

3.1 Logical Possibility with Subscripts

Let me begin by precisifying the basic notion of logical possibility (denoted by $\Diamond$) at issue here. To evaluate whether a claim ϕ requires something logically possible (in this sense), we hold fixed the operation of logical vocabulary (like $\exists, \wedge, \vee, \neg$), but abstract away from any further metaphysically necessary constraints on the application of particular relation symbols. Thus, we consider all possible ways for relations to apply (including those ways that aren't definable). For example, it is logically possible that $(\exists x)(\text{Raven}(x) \wedge \text{Vegetable}(x))$, even if it would be metaphysically impossible for anything to be both a raven and a vegetable. We also abstract away from constraints on the size of the universe[1], so that $\Diamond(\exists x)(\exists y)(\neg x = y)$ would be true even if the actual universe contained only a single object. Note that this notion of logical possibility is not defined in terms of mere syntactic consistency within some formal deduction system.

Philosophers advocating a range of different philosophies of mathematics have invoked a similar notion.[2] This notion of logical possibility corresponds to our intuitive sense that certain descriptions of structures (like second order Peano Arithmetic[3]) require something coherent, while others (like Frege's

[1] See Etchemendy's [2] on the tension between standard Tarskian reinterpretation-based accounts of logical possibility and the intuitive notion of logical possibility regarding this point.

[2] maybe cite: Hartry Field, Shapiro, Rayo

[3] Note, however, that to assert a version of second-order Peano Arithmetic we will need to use relativized logical possibility, as we will see below.

inconsistent theory of extensions) do not.

I think we can also intuitively understand claims about logical possibility 'given' the facts about how certain relations apply. Consider a statement like the following.

> Given what cats and blankets there are, it is logically impossible that each cat slept on a different blanket last night.

This sentence has an intuitive reading which employs a notion of logical possibility *holding fixed the way that certain relations apply* (in this case, holding fixed what cats and blankets there are) rather than logical possibility *simpliciter*. A moment's thought will reveal that (on this reading) the above sentence is true if and only if there are more cats than blankets.

I propose to think of the logical possibility $\Diamond_{(...)}(...)$ as an operator which takes a sentence ϕ and a finite (potentially empty) list of relation symbols $R_1, ...R_n$ and produces a sentence $\Diamond_{R_1,...R_n}\phi$ which says that it is logically possible for ϕ to be true, without any change to how the relations $R_1, ...R_n$ apply. Thus, for example, the claim, 'Given what cats and baskets there are, it is logically impossible that each cat slept in a distinct basket' becomes:

$$\mathbf{C}\wedge\mathbf{B}{:}\neg\Diamond_{cat,basket}[(\forall x)(cat(x) \to (\exists y)(basket(y)\wedge sleptIn(x,y)\wedge (\forall z)[cat(z) \wedge sleptIn(z,y) \to x = z])]$$

Finally, note that by using this notion we can also make *nested* logical possibility claims, i.e., claims about the logical possibility of scenarios which are themselves described in terms of logical possibility. I have in mind sentences like the following:

$\Diamond$**C**$\wedge$**B**: $\Diamond(\neg\Diamond_{cat,basket}[(\forall x)(cat(x) \to (\exists y)(basket(y) \wedge sleptIn(x,y) \wedge (\forall z)[cat(z) \wedge sleptIn(z,y) \to x = z])])$

The above sentence, $\Diamond$(C$\wedge$B), expresses a truth because (reading from the outside in):

- It is logically possible (holding fixed nothing) that there are 4 cats and 3 baskets.
- Relative to the logically possible scenario where there are 4 cats and 3 baskets, it is not logically possible (given what cats and baskets there are), that each cat slept in a basket and no two cats slept in the same basket.

Based on these kind of examples, I take logical possibility sentences of the form $\Diamond_{R_1 \ldots R_n}\phi$ to be meaningful, even in cases where ϕ is itself a sentence which makes appeal to facts about logical possibility. As noted above, I will not allow sentences which quantify in to the $\Diamond$ of logical possibility.

To clearly express claims about logical possibility, we can define a formal language $\mathcal{L}$, which I will call the language of logical possibility (though no implication that this exhausts the concept should be drawn). Fix some infinite collection of variables and relation symbols of every arity together with $\bot$ and define $\mathcal{L}$ to be the smallest language built from these variables using these relation symbols and equality closed under applications of the normal first order connectives and quantifiers and $\Diamond_{\ldots}$ (where $\Diamond_{\ldots}$ expressions can only be applied to sentences (so there is no quantifying in). We will also use $\Box_{\ldots}$ in our sentences but regard it as an abbreviation for $\neg \Diamond_{\ldots} \neg$

3.2 Contrast With Other Modal Notions

Before going on, it may help philosophical readers to note how my notion of logical possibility differs from three vaguely similar modal notions in the literature (Tarskian re-interpretiability, metaphysical possibility and conceptual possibility) as follows.

The notion of logical possibility is (potentially) less demanding than the notion of Tarskian reinterpretability, for reasons discussed in Etchemendy's *The Concept Of Logical Consequence*. Essentially, the issue is that certain scenarios might be genuinely logically possible but require the existence of more objects than actually exist, and hence not permit any Tarskian reinterpretation (since Tarskian reinterpretations of a sentence must still take the sentence's quantifiers to range over some collection of objects in the actual world).

The notion of logical possibility is strictly less demanding than the notion of metaphysical possibility. For, as Frege noted, the laws of logic hold at all possible worlds. Yet (as noted above) statements like $(\exists x) Raven(x) \wedge Vegetable(x)$[4] can require something which is logically possible but metaphysically impossible.

Finally, the notion of logical possibility is also strictly less demanding than the notion(s) of idealized conceivability and/or conceptual possibility which occur in debates over philosophical zombies and Chalmers' *Constructing the World* (and are, inconveniently, sometimes also labeled logical possibility). For the notion of conceptual possibility reflects something like ideal a priori

[4] I won't get into debates about what the true logical form of non-mathematical natural language sentences like 'something is both a raven and a vegetable' here.

acceptability, so that when evaluating whether it is conceptually possible that ϕ we have to preserve all analytic truths associated with relations occurring in ϕ. In contrast (as I have noted above) logical possibility abstracts away from all such specific features of relations. Thus, for example, if it is analytic that $(\forall x)(bachelor(x) \rightarrow male(x))$, then it will be logically possible but *not* conceptually possible that $(\exists x)(bachelor(x) \wedge \neg male(x))$.