Chapter 5

The Formal System II: Other Inference Rules

(Cut) Cutback If $\mathcal{L} = R_1, \ldots, R_m$ is a list of relations and $\vec{x}^i = x_1^i ... x_{n_i}^i$ where n_i is the arity of R_i, then $\Gamma \vdash (\exists x)P(x) \wedge (\forall \vec{x}^1)(R_1(\vec{x}^1) \rightarrow P(x_1^1) \wedge ... P(x_{n_1}^1)) \wedge ... (\forall \vec{x}^m)(R_m(\vec{x}^m) \rightarrow P(x_1^m) \wedge ... P(x_{n_m}^m)) \rightarrow \Diamond_{\mathcal{L}}(\forall x)P(x)$

This axiom schema expresses the idea that if a predicate P applies to all the objects which relations in $\mathcal{L}$ apply to (and P applies to at least one thing), then it is logically possible (given the facts about what P applies to and about the relations in $\mathcal{L}$) that P applies to the whole universe.

(ReL) Relabeling. If $R_1 \ldots R_n$ are relations that occur in θ but not in $\mathcal{L}$, and $R_1' \ldots R_n'$ are relations with the same arities (i.e., the arity of R_i and R_i' are the same) that don't occur in $\mathcal{L}$ or θ, then $\Gamma \vdash \Diamond_{\mathcal{L}}\theta \leftrightarrow \Diamond_{\mathcal{L}}\theta[R_1/R_1' \ldots R_n/R_n']$.

This axiom schema expresses the idea that when evaluating claims about logical possibility, all relations of the same arity have the same behavior (we abstract away from their underlying behavior). Thus, replacing some $R \notin \mathcal{L}$ with an unused relation $R' \notin \mathcal{L}$ of the same arity cannot change the truthvalue of $\Diamond_{\mathcal{L}}\theta$.

Example: By substituting sleeps with chews we see "It is logically possible, given the facts about dogs and blankets, that every dog sleeps on a different blanket" $\Leftrightarrow$ "It is logically possible, given the facts about dogs and blankets, that every dog chews on a different blanket."

Note that sleeps and chews are both relations that are not in the list of relations being subscripted $\mathcal{L}$ = dog, blanket.

(SC) Simple Comprehension. If ψ is a sentence which contains no $\Box$s or $\Diamond$s and the relation R doesn't occur in $\mathcal{L}, \phi$ or ψ, then $\Gamma \vdash \psi \to \Diamond_{\mathcal{L}}[\psi \wedge (\forall \vec{z})(R(\vec{z}) \leftrightarrow \phi(\vec{z}))]$.

This axiom schema captures the idea that it is possible (holding fixed $\mathcal{L}$) for an otherwise unused relation R to apply to exactly those tuples $\vec{z}$ which satisfy some first order formula $\phi(\vec{z})$. Moreover, intuitively it is possible for R to be so defined without changing the truth of any sentences not containing R.

> Example: "If there is something which everyone loves, it is logically possible (given the facts about love) that there is something which everyone loves *and* happy() applies to exactly those individuals which love themselves."

Our next axiom schema, Modal Comprehension, expresses a somewhat similar idea to the Simple Comprehension Schema above. It says that one can sometimes specify a logically possible way for a relation R to apply by appealing to properties that can only be expressed using modal operators, e.g., the property of x envying infinitely many objects.

Informally, we can specify how an (otherwise unused) relation R applies by saying that it applies to exactly those n-tuples of objects in $Ext(\mathscr{L})$, which satisfy a certain property expressed in terms of logical possibility operators and the relations in $\mathscr{L}$. It lets us express (and recognize the truth of) claims which seem to require quantifying in, like:

> SIBLINGS: Holding fixed the facts about the relations Married(x, y) and Sibling(x, y) it is logically possible to have a relation $R(x)$ that applies to exactly those married individuals x with more siblings than their spouse.

Note that having more siblings than one's spouse has to be cashed out in terms of the logical possibility of a surjective but not injective map from their siblings to those of their spouse. On first glance, it would appear this would require passing x (the individual for whom we wish to compare their siblings to those of their spouse) into the logical possibility operator evaluating the possibility of such a pairing. However, our language of logical possibility does not allow this kind of quantifying in.

Instead, we do this by using a special, otherwise-unused, n-place relation Q to label and preserve a choice for an n-tuple of objects in $Ext(\mathscr{L})$. We say that it is possible (fixing the $\mathscr{L}$ facts) for R to apply in such a way that, necessarily (fixing the $\mathscr{L}, R$ facts), R only relates objects in $Ext(\mathscr{L})$ and however Q chooses a unique n-tuple of objects in $Ext(\mathscr{L})$ for consideration, R applies to this n-tuple iff a certain modal claim ϕ describing the behavior of $\mathscr{L}$ and Q is true. In this case, the relevant $\mathscr{L}$ is Married, Sibling, and the modal sentence ϕ is $\Diamond_{\text{Married,Sibling},Q}$ $((\exists x)Q(x) \wedge (\exists y)\text{Married}(x,y)$ and $Z(\cdot,\cdot)$ is a surjective but not injective map from the siblings of x to those of y.

We can thus express the informal claims like siblings

$$\begin{aligned} &\Diamond_{\mathscr{L}} \Box_{\mathscr{L},R}(\exists! x Q(x) \rightarrow \\ &\exists x(Q(x) \wedge [R(x) \leftrightarrow x \in Ext(\mathscr{L}) \wedge \phi)] \end{aligned} \tag{5.1}$$

Since it is possible for Q to apply to any single object the necessity operator above ensures that R applies to exactly those x which have more siblings than their spouse. With this motivation in place, I can now state the Modal Comprehension Schema as follows

(MC) Modal Comprehension If

- R does not occur in $\mathcal{L}, \psi$ or ϕ
- Q does not occur in $\mathcal{L}$ or ψ
- ϕ is content restricted to $\mathcal{L}, Q$

then $\Gamma \vdash \psi \rightarrow \Diamond_{\mathcal{L}}(\psi \wedge \Box_{\mathcal{L},R}(\exists! \vec{x} Q(\vec{x}) \rightarrow (\exists \vec{x})(Q(\vec{x}) \wedge [R(\vec{x}) \leftrightarrow \vec{x} \in Ext(\mathscr{L}) \wedge \phi])))]$

where $\exists! \vec{x} Q(\vec{x}$ means that Q applies to a unique n-tuple of objects.

(Inf) Infinity It is possible for a two place relation S to apply in the following successor-like way:

- The successor of an object is unique $(\forall x)(\forall y)(\forall y')[S(x,y) \wedge S(x,y') \to y = y']$
- successor is one-to-one $(\forall x)(\forall y)(\forall x')(S(x,y) \wedge S(x',y) \to x = x')$
- there is a unique object that has a successor and isn't the successor of anything $(\exists! x)(\exists y)\,(S(x,y) \wedge (\forall y)\,[\neg S(y,x)])$
- everything that is a successor has a successor $(\forall x)[(\exists y)S(y,x) \to (\exists z)S(x,z)]$
- S is anti-reflexive: $(\forall x)(\forall y)[S(x,y) \to \neg S(y,x)]$

(PP) Possible Powerset. If F, C are distinct predicates and $\in$ a two-place relation, then $\Gamma \vdash \Diamond_F \mathscr{C}(C, \in, F)$.

Here $\mathscr{C}(C, \in, F)$ means that C and F are disjoint, $\in$ relates (only) objects satisfying C to objects satisfying F and:

- $\Box_{C,\in,F}(\exists x)[C(x) \wedge (\forall y)((F(y) \wedge K(y)) \leftrightarrow y \in x)]$, i.e., it's necessary that however some new predicate K applies to some objects satisfying F, there exists a corresponding 'class' C whose 'elements' are exactly the objects which F applies to.
- $(\forall y)(\forall y')(C(y) \wedge C(y') \wedge \neg y = y' \to (\exists x)\neg(x \in y \leftrightarrow x \in y')$, i.e., no two members of C contain (in the sense of $\in$) the same elements.

Intuitively, this axiom schema says that it is always possible to "add a layer of classes" to the objects satisfying some predicate F.

(Choice) Combinatorial Choice. $\Gamma \vdash (\forall x)[I(x) \to (\exists y)R(x,y)] \to \Diamond_{I,R}[(\forall x)(\forall y)(\hat{R}(x,y) \to R(x,y)) \wedge [(\forall x)(I(x) \to (\exists! y)\hat{R}(x,y)]])$

This axiom schema captures the same intuition as the axiom of choice in set theory. It says that if every x satisfying I is related to some y by R, then (fixing I, R) another relation $\hat{R}$ can behave like a choice function selecting a unique such y for each x.

(CR) Combinatorial Replacement (aka the Chia Pet Axiom Schema) If

- $\mathcal{L}$ is a list which contains the predicate I but not P
- ϕ is content-restricted $\mathcal{L}, P, R_1 \ldots R_n$. (where P, $R_1 \ldots R_n$ and $\mathcal{L}$ share no relations)
- $\hat{R}_1 \ldots \hat{R}_n$ are otherwise unused relations such that if R_i is an n-place relation $\hat{R}_i$ is an $n+1$ place relation.

Let $\rho(x,y)$ be the following formula

$$\bigvee_{\substack{1 \le i \le n \\ 1 \le j \le l_i}} (\exists z_1)\ldots,(\exists z_{j-1}),(\exists z_{j+1}),\ldots,(\exists z_{l_i})\hat{R}_i(z_1,\ldots,z_{j-1},x,z_{j+1},\ldots,z_{l_i},y)$$

In other words $\rho(x,y)$ asserts that x appears in some tuple ending with y satisfying some $\hat{R}_i$

Let $\Psi(x)$ be the formula

$$\bigwedge_{1 \leq i \leq n} (\forall \vec{v})(R_i(\vec{v}) \leftrightarrow \hat{R}_i(\vec{v}, x))$$

asserting that $\hat{R}_i$ with x inserted into the last place behaves exactly the same as R_i

then

$$\begin{aligned} &\Gamma \vdash \Box_{\mathcal{L}}[\exists! x(P(x) \wedge I(x)) \rightarrow \Diamond_{\mathcal{L},P} \phi] \rightarrow \\ &\quad \Diamond_{\mathcal{L}} \Big[(\forall x)(\forall y)(\forall y') \Big[(\neg y = y' \wedge \rho(x,y) \wedge \rho(x,y') \rightarrow x \in Ext(\mathcal{L}) \Big] \wedge \\ &\qquad \Box_{\mathcal{L}, \hat{R}_1 \ldots \hat{R}_n} [\exists! x(P(x) \wedge I(x)) \wedge \exists x(P(x) \wedge I(x) \wedge \Psi(x)) \rightarrow \phi \Big] \end{aligned}$$

Crudely speaking, this principle takes us from the logical possibility (given facts about $\mathcal{L}$), of satisfying a certain formula $\phi(x)$ *for any single* x in a base collection of objects (those satisfying I) by creating a suitable miniature universe around this object, to the logical possibility of simultaneously extending the universe so that *for every* object x in this base collection, there is a corresponding miniature universe (indexed to this object x) in which $\phi(x)$ is satisfied.

My final principle says that if it's possible for each "seed",i.e., index object satisfying I, there can be a "sprout", i.e., miniature universe satisfying some property, then one could have a scenario in which all of these seeds sprout simultaniously so one has a full hairy chia pet (i.e. a universe in which the application of U' and $R'_1 ... R'_n$ code up the behavior of suitable miniature universes around each object satisfying I.

For example, if we take I to be the predicate $\mathrm{person}(\cdot)$ and the $\mathcal{L}$ to be the list $\mathrm{person}(\cdot), \mathrm{childOf}(x, y)$

> If, for any choice of a person, there could be as many ghosts as that person has children, then it could be that for every person x there are as many ghosts for x (disjoint from everyone else's ghosts) as they have children

As before, articulating this principle can seem to require quantifying in to the $\Diamond$ of logical possibility. However, we can use the same trick as before of subscripting relations instead of quantifying in.