Chapter 6

Example: Lemmas about Well-Orderings

To give a more visceral sense of how the above axioms work, I will now present two proofs in my system of logical possibility that that mirror results in set theory (which can be found in elementary texts like [4]).

In later chapters I will present proofs in a more informal style. However I will try to make it clear to the reader how each of these proofs can be expanded out into the fully formal proofs demonstrated below.

In what follows, I will sometimes use $\exists_F x\phi(x)$ to abbreviate $\exists x F(x) \wedge \phi$ and $\forall_F x\phi(x)$ to abbreviate $(\forall x)(F(x) \rightarrow \phi(x))$.

6.1 Lemma A

Jech's version of the first lemma I am going to prove says the following:

"If $(W, \leq)$ is a well-ordered set and $f : W \rightarrow W$ is an increasing function, then $x \leq f(x)$ for each $x \in W$."

We can express something like the same idea in the language of logical possibility by making the following definitions:

Definition A two-place relation $\leq$ **well-orders** the objects which satisfy F iff

- $\leq$ linearly orders the objects which satisfy F
  - $\leq$ is anti-symmetric

    $(\forall x)(\forall y)(x \leq y \wedge y \leq x \rightarrow x = y)$
  - $\leq$ is transitive

    $(\forall x)(\forall y)(\forall z)(x \leq y \wedge y \leq z \rightarrow x \leq z)$
  - $\leq$ is total on F

    $(\forall x)(\forall y)(F(x) \wedge F(y) \rightarrow x \leq y \vee y \leq x)$
- F, $\leq$ satisfy the Least Element Condition: If some element satisfying F also satisfies G then there is a least element in F satisfying G:

  $\Box_{F,<}[(\exists x)(F(x) \wedge G(x)) \rightarrow (\exists y)(F(y) \wedge G(y) \wedge (\forall z)(F(z) \wedge G(z) \rightarrow y \leq z))]$

Definition A two place relation R **behaves like an increasing function** from some $F, \leq$ to $G, \leq'$ if

- $(\forall x)[F(x) \rightarrow (\exists y)(G(y) \wedge R(x,y))]$
- $(\forall x)(\forall y)[R(x,y) \rightarrow (\forall z)(R(x,z) \rightarrow y = z)]$

- $(\forall x)(\forall y)(\forall x')(\forall y')[x \leq y \wedge R(x,x') \wedge R(y,y') \rightarrow x' \leq y']$.

I will sometimes abbreviate the above claim as $F \leq \mapsto_{R,inc} G \leq'$ since it says that R behaves like an increasing function which maps from $F \leq$ (the Fs under $\leq$) to $G \leq'$ (the Gs under $\leq'$).

As usual, I will sometimes abbreviate the claim that $x \leq y \wedge \neg x = y$ as $x < y$.

Thus we can write a version of Jech's Lemma follows:

> **Lemma A:** "If R behaves like an increasing function from $F, \leq$ to $F, \leq$ and the objects satisfying F are well ordered by $\leq$, then $(\forall x)(\forall y)[F(x) \wedge F(y) \wedge R(x,y) \rightarrow \neg y < x]$ "

Proof. To prove Lemma A, we will use essentially the same reasoning which Jech uses to prove his set theoretic version of this claim. He says: Let F be well-ordered by $<$ and let R behave like an increasing function from F to F $<$. Suppose the conclusion of the lemma does not hold. Then by the fact that F is well-ordered by $<$, there must be a least x such that $f(x) < x$, (i.e. $(\exists y)(R(x,y) \wedge y < x)$. Consider $y = f(x)$. Because $y < x$ and R is increasing, $f(y) < f(x) = y$. This contradicts the claim that x is the $\leq$-least z such that $F(z) \wedge f(z) < z$.

To mimic this reasoning, we first suppose that $<$ well-orders the Fs and R behaves like an increasing function from the Fs to the Fs. Now we want to consider the collection of objects satisfying F which R maps below themselves (just as in Jech's proof). Our main difficulty will be in using the fact that the Fs are well-ordered by $\leq$ (in the sense specified by the modal definition of well ordering above) to show that if there are any such objects, there must be an $\leq$-least one.

By Simple Comprehension it is possible (while holding fixed the facts about how the relations F, R and $\leq$ indicated in curly braces apply) for the otherwise-unused predicate G to apply to exactly those objects satisfying F and which R maps below themselves. Within this context, our initial assumptions that $<$ well-orders the Fs and R behaves like an increasing function from the Fs to the Fs must remain true (because they are content restricted to F, R and $\leq$). So we can use the fact that the $F, \leq$ satisfy the Least Element condition, i.e. $\Box_{F,\leq}$(if G applies to any object satisfying F, it applies to an $\leq$-least such object) to deduce the needed claim. So we have: if R maps any object satisfying F below itself, then then G applies to exactly these objects, and there is a $\leq$-least object satisfying G.

Once we have this fact, we can derive the truth of the desired claim that $(\forall x)(\forall y)[F(x) \wedge F(y) \wedge R(x,y) \rightarrow \neg y < x]$ holds true *within this special context* by exactly the same first order reasoning which Jech uses. Suppose for contradiction that there were some such x that got mapped below itself (in the sense above). Then we'd have a y such that $R(x,y) \wedge y < x$. By the fact that R is a one-to-one increasing function, $y < x$ and R maps x to y, we know that R must map y to something strictly less than y[1]. But then G must also apply to y, which contradicts the claim that x is the $\leq$-least object satisfying G.

Thus we know that $(\forall x)(\forall y)[F(x) \wedge F(y) \wedge R(x,y) \rightarrow \neg y < x]$ holds true

[1] It must map y to something $\leq y$ because it is increasing and something $\neq y$ because it is one-to-one and $R(x,y)$ and $\neg x = y$.

within the special modal scenario being considered above. However, we can note the above claim is content-restricted to $F, R, \leq$. Thus we can infer from the mere fact that it could be true (while holding fixed the behavior of F, R and $\leq$) to the conclusion that it is actually true.

Now the desired conclusion, and the overall conditional to be proved follows immediately.

1		$<$ well-orders the $Fs \wedge F \leq\mapsto_{R,inc} F \leq$	[1]
2		$\Diamond_{F,\leq,R} \forall z[G(z) \leftrightarrow \exists_F y R(z,y) \wedge y < z]$	SC
3	$\Diamond$ |	$\forall z[G(z) \leftrightarrow \exists_F y R(z,y) \wedge y < z] \quad \{F, \leq, R\}$	In$\Diamond$ I
4	|	$<$ well-orders the $Fs \wedge F <\mapsto_{R,inc} F \leq)$	2 import [1]
5	|	$\Box_{F,\leq}(\exists_F x G(x) \rightarrow \exists_F x[G(x) \wedge (\forall_F z G(z) \rightarrow x \leq z)])$	4 FOL [1]
6	|	$\exists x(F(x) \wedge G(x)) \rightarrow (\exists_F x[G(x) \wedge (\forall_F z G(z) \rightarrow x \leq z)]$	5 $\Box$ E [1]
7	|	$\neg[\exists_F x \exists_F y(R(x,y) \wedge y < x)]$	3,4,6 FOL [1,4]
8		$\Diamond_{F,\leq,R} \neg[\exists_F x \exists_F y R(x,y) \wedge y < x]$	2-7 In$\Diamond$E [1]
9		$\neg[\exists_F x \exists_F y R(x,y) \wedge y < x]$	8 $\Diamond$E [1]
10		$\forall_F x \forall_F y(R(x,y) \rightarrow \neg y < x)$	9 FOL [1]
11		$\forall_F x \exists_F y R(x,y) \wedge (y = x \vee y > x)$	1, 10 FOL [1]

□

6.2 Lemma B

"No well-ordered set is isomorphic to an initial segment of itself"

Definition I will say that **the** $R_1, \ldots R_m$ **are isomorphic to the** $\langle R'_1, \ldots R'_m \rangle$ **under some relation Z** (henceforth written $\langle R_1, \ldots R_m \rangle \cong_Z \langle R'_1, \ldots R'_m \rangle$) iff:

- Z behaves like a bijection between the domain of objects in $Ext(R_1, \ldots R_m)$ and the domain of objects in $Ext(R'_1, \ldots R'_n)$. More formally:
  - $(\forall x)(x \in Ext(R_1, \ldots R_m) \rightarrow (\exists! y s.t. Z(x,y)) \wedge y \in Ext(R'_1, \ldots R'_m)])$ i.e., Z is functional over $Ext(R_1, \ldots R_m)$ with a range within $Ext(R'_1, \ldots R'_m)$ and
  - $(\forall y)(y \in Ext(R'_1, \ldots R'_m) \rightarrow (\exists! x s.t. Z(x,y) \wedge y \in Ext(R_1, \ldots R_m)]),$ i.e., Z maps one-to-one and onto all of $Ext(R'_1, \ldots R'_m)$
- Z applies in a way that respects each R_i, i.e., $(\forall \vec{x})(\forall \vec{y})[Z(x_1, y_1) \wedge \ldots \wedge Z(x_n, y_n) \rightarrow (R_i(\vec{x}_i)) \leftrightarrow R'_i(\vec{y}_i))]$, where if R_i is an n-place relation then $\vec{x}_i = x_1, \ldots x_n$ and $\vec{y}_i = y_1, \ldots y_n$

We can then state a modal version of this claim as follows.

Claim to Prove: If the objects satisfying F are well-ordered by $\leq$ then $\neg \Diamond_{F,\leq} (\exists x)[F(x) \wedge \langle F; > \rangle \cong_R \langle G; > \rangle \wedge \forall z (G(z) \leftrightarrow [F(z) \wedge z < x])]$

Proof. Assume that the objects satisfying F are well-ordered by $\leq$. Suppose for contradiction that $\Diamond_{F,<}[(\exists x)(F(x) \wedge (\forall z)(G(z) \leftrightarrow F(z) \wedge z < x) \wedge \langle F, > \rangle \cong_R \langle G, > \rangle)]$. Consider any such scenario. The fact that the objects satisfying F are well-ordered by $<$ must remain true in this scenario (because it is content-restricted to F, G, R and $>$). By first order logic and unpacking definitions we can deduce that R therefore behaves like an increasing function from $F, \leq$ to $F, \leq$ (the key fact is that R must respect $\leq$).

Now, to get contradiction, we can copy over Lemma A (we have just seen that it can be proved from empty premises, so can we re-prove it as needed, within any $\Box$ or $\Diamond$ context) and derive that R does not map any object satisfying F strictly below itself. On the other hand, we know there is an object x satisfying F which is $>$ all objects satisfying G and that $\langle F, > \rangle \cong_R \langle G, > \rangle$. It follows from this by simple first order logic that R maps the any such x to a some object $y < x$. Thus contradiction/the false ($\perp$) would have to obtain in the (supposedly) logically possible scenario under consideration.

Finally, we can export this $\perp$ to our original situation (remembering that the contradiction symbol $\perp$ is content-restricted to every list of relations) and thereby complete our proof. Informally, this corresponds to reasoning that if it were $\Diamond_{F,<}[(\exists x)(F(x) \wedge (\forall z)(G(z) \leftrightarrow F(z) \wedge z < x) \wedge \langle F, > \rangle \cong_R \langle G, > \rangle)]$ then it would also be $\Diamond_{F,<}\perp$, which is false, so the original $\Diamond_{F,<}$ claim cannot be true.

1		F is well-ordered by $<$	[1]
2		$\Diamond_{F,<}[(\exists x)(F(x) \wedge (\forall z)(G(z) \leftrightarrow F(z) \wedge z < x) \wedge \langle F; > \rangle \cong_R \langle G; > \rangle)]$	[2]
3	$\Diamond$	$(\exists x)(F(x) \wedge (\forall z)(G(z) \leftrightarrow F(z) \wedge z < x) \wedge \langle F; > \rangle \cong_R \langle G; > \rangle)\{F, <\}$	In$\Diamond$I [2]
4		F is well-ordered by $<$	1 import [1]
5		R behaves like an increasing function on F	4 FOL [2]
6		Well-Ord.: F $< \wedge R$ Inc. Func.:$F \rightarrow [\forall_F x \forall_F y R(x, y) \rightarrow \neg y < x]$	lemma A
7		$\forall_F x \forall_F y (R(x, y) \rightarrow \neg y < x)$	6,7,8 FOL [1,2]
8		$(\exists y)(R(x, y) \wedge G(y))$	5 FOL [5]
9		$(\exists x)(\exists y)(F(x) \wedge F(y) \wedge R(x, y) \wedge y < x)$	5,10 FOL [2,5]
10		$\bot$	9,11 FOL[1,2]
11		$\Diamond_{F,<}(\bot)$	2,3-10 In$\Diamond$E [1
12		$\bot$	11 $\Diamond$E [1,2]
13		$\neg \Diamond_{F,<} [(\exists x)(F(x) \wedge (\forall z)(G(z) \leftrightarrow F(z) \wedge z < x) \wedge \langle F, > \rangle \cong_R \langle G, > \rangle)]$	$\neg$I [1,2]

So $\vdash$ F is well-ordered by $< \rightarrow \neg \Diamond_{F,<} [\exists x F(x) \wedge \Diamond_{F,<} \langle F; > \rangle \cong_R \langle G; > \rangle \wedge \forall z (G(z) \leftrightarrow [F(z) \wedge z < x])]$

□