A Foundation For Potentialist Set Theory

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Part I

A Carnapian Explication of Set Theory

Chapter 1

Introduction

1.1 My aim

Set theory is widely accepted as a foundation for all of modern mathematics. And it is hard to deny that the mathematical results which are currently stated in terms of set theory constitute powerful and important knowledge of some kind.

Yet some vexing philosophical puzzles remain about how to understand set theory. General worries have been raised concerning all mathematical objects (e.g., materialist/empiricist objections existence of abstract objects, and access/Benacerraf worries about how human accuracy about mathematics could be anything but a miracle or a mystery). But even if we lay these worries aside, there are specific puzzles in the philosophy of set theory -about whether we have a non-paradoxical conception of the height of the hierarchy of sets, and how to justify use of the axiom of replacement - to which no answer is widely accepted.

[For example the most central set theory specific problem which motivates my work in this book (and the development of potentialist set theory in general) goes like this. We seem to reason about the hierarchy of sets using bivalence as though we had a definite structure in mind, just as we do when we talk about the natural numbers. But if we try to articulate what this structure is, we can describe the width of the heirarchy of sets fairly well (it is build up in layers that are well ordered, and every successor layer contains the powerset of what's below, and every limit layer contains the union of all the sets below), but we cannot say anything equally definite about the height. We take the height to be some limit stage (i.e., we think there is no last layer), and for there to be infinitely many layers, and for axioms like replacement to hold within the total hierarchy of sets (which implies certain constraints on the height, especially in conjunction with the description of the width of the sets just noted). But this conception of the sets doesn't seem to uniquely determine the height of the hierarchy of the sets. It is generally thought that there are multiple logically coherent and non-isomprhic structures (V up to some inaccessible ordinal) which each satisfy all these constraints.

Naively, one might like to say that our conception of the hierarchy of sets is that it 'goes all the way up', in the sense that for any way that some objects are (or could be) well ordered, the hierarchy of sets goes up that far. From this point of view all the claims about the layers of sets being infinite, having no last layer, and satisfying replacement will all be correlaries of our base conception of absolute infinity. But as the Burelli-Forti paradox points out this naive conception is paradoxical. If you take the layers in the hierarchy of sets (or the sets called ordinals corresponding to them) and consider the well ordering which places a piece of cheese above all ordinals, we have a possible (and indeed actual) way that some objects are well ordered which surpasses all the layers in the hierarchy of sets.

Thus we don't seem to have a consistent conception of the structure of the heirarchy of sets which specifies how high up it is supposed to go. If we think about the sets in a straightforward actualist way, analogous to how we think about cities and numbers, then it appears we must say that the hierarchy of sets just stops at some point, though there are actual well orderings (i.e., relations that we can define) which go up higher, and it would be intuitively quite coherent and logically possible for the heirarchy of sets to go up that further level. Thus accepting an actualist approach to set theory (once we have renounced the paradoxical conception of sets torpedoed by the Buralli-Forti paradox) seems to commit us to a kind of deep arbitrariness in mathematical reality. We have to say that the hierarchy of sets just happens to stop at some point, such that it would be quite coherent for it to go up further (taking another limit stage), and indeed (as Wright and Shapiro helpfully put it) it seems that all reasons we have for thinking that there are sets at all seem to do an equally good job of motivating the claim that the sets go past this point.

In this book I will try to solve this and other set theory specific puzzles, by a combination of philosophical and mathematical work. Philosophically, I will develop and argue for a certain Carnapian explication of set theoretic talk in potentailist terms. That is, I will argue that (even mathematicians our current understanding of set theory is vague or different from my proposal in some respects) we can best avoid/dissolve various set theory specific puzzles by understanding set theory talk potentialistically, as making a certain kind of modal claims about logical possibility and extendability.

Mathematically, I will provide a formal foundation for set theory so understood. That is, I will provide an intuitively compelling a formal system for reasoning about logical possibility, and show that it is sufficient to reconstruct main stream matheamticians set theoretic reasoning (i.e., the use of first order logic plus the standard Zermelo Frankel plus Choice axioms of set theory).

1.2 On the relevant notion of Carnapian Explication

To clarify the kind of Carnapian Explication I am attempting, it may be helpful to consider a dilemma posed by Burgess and Rosen for nominalists about mathematical objects in general who (like me, but for rather different reasons) suggests that we can/should think about mathematical claims which appear to have one logical form as really having another. The dilemma is: is one claiming that mathematicians do mean the propositions put forward in the new philosophical paraphrases, or that they should? If one is making the hermenutical claim that they do, this seems rather unchritable given the diversity of philosophical positions including many rejctions of nomainlism/potentialis which mathmaticians have expressed. If one things that they should, how can one account for the fact that a paper merely proposing nominalistic reformulation of a standard physical theory/a set theoretic theorem would probably be rejected by top physics/mathematics journals as not contributing something (and doesn't it seem rather bold for philosophers to be telling physicsists/mathematicians working in such apparently more fruitufl domains that they should be acting differently.

I respond to this challenge by essentially embracing the revisionary horn of the dilemma (as suggested by the idea of carnapian explicatiosn being pragmatically better successor concepts), with two important caveats which I will now note.

I'm suggesting potentialist paraphrases are what people should mean when they do set theory. They "should" replace current set theory with the potentialist version because understanding things potentialistically blocks various intuitive puzzles, and makes sense of thigns that we normally want to say about set theory (maybe even ways we want to use set theory?). In many ways the project of developing potentailist foundation for set is analogous to the project of providing a set theoretic foundation for analysis. Our naive reasoning about certain concepts (limits in one case, the height of the hierarchy of sets in the other) turned out to lead to paradox. So it is desirable to find a different way of thinking about/sharpening/explicating the relevant mathematical concepts which will let us capture the intuitive mathematical significance and interest of set theoretic/number theoreitc claims while blocking paradoxical inferences, and providing cashing out old mathematical concepts which the paradoxes may have lead us to doubt that we have a coherent grip on in other terms, which we seem to understand in a way that does not invite paradox.

[discuss whether I should keep this caveat, or keep the interesting philosophical upshots of it being the best explication belwo]However (first caveat) I don't claim to show that the potentailist formulation is the only possible formulation which secures the pragmatic benefits above. I'll suggest that this way of banishing intuitive paradoxes is better than all the options in the current literature. But perhaps there is some different but equally good successor to current set theoretic practices which banishes intuitive puzzlement in a very different way yet to be discovered. Although standard set theoretic definitions/paraphrases of claims about limit, continuity etc. in Calculous provide one good carnapian explication/successor for Newton and Leibnitz' practice, nonstandard analysis (which takes the idea of infinitessimals seriously) provides a different explication which banishes the same paradoxes of infinity and might be equally philosophically coherent and matheamticaly intersting to study. So I don't claim that my propsed way of thinking about set theory/sharpening ordinary set theoretic concepts is the only possible way, or uniquely reference magnetic in some way that makes it what people meant all along.

Also (second caveat) I don't claim set theorists should literally rewrite textbooks and journal articles in terms of the rather cumbersome notation of boxes and diamonds which I will introduce in this pick. Current practice of just proving things from ZFC is fine, and our minds work better if we just have to do first order logical deductions vs. thinking about elaborate modal extendability claims. I'm suggesting that we think about first order claims as abbreviating claims about boxes and diamonds for the purposes of wondering about the meaning of set theory. The main result in this book attempts to show that it is harmless to make this abbreviation.

So what do I say to Burgess and Rosen's challenge for revisionists: how I can say that such potentailist reformulation is an improvement if mathematical journals wouldn't accept it? Although I think it would be better for people to start talking about potentialist set theory (in pure mathematical contexts, using set theoretic talk as an abbreviation), I admit that set theory journals would probably not treat this reformulation as an advance, and certainly not publish it. But I think this fact is quite compatible with potentialist sharpening of set theory being ideally better, and reflects a pragamatically good (given the reality of philosophical disagreements and the diversity of philosophical views among mathematicians) feature of (axiomatic) mathematical practice, which we have separate reason for accepting. The feature of mathematics I have in mind is Kenny Easawarn's point that mathematical axioms just articulate points of consensus, which can be common knowledge and taken for granted within mainstream mathematics. There's a kind of useful modularity to axiomatic mathematics. Mathematicians can (and do) understand the axioms as meaning radically different thigns, or nothing at all (c.f. the saying that formalism is the sunday religion of mathematics), but still usefully work together on proving things from them. Given widespread intra-matheamtical disagreement on what to say beyond the axioms about the nature of sets and elementhood etc, its good for journals just to proceed with the concepts mentioned in the axioms. Each person (platonist, fictionalist, formalist, category theoretic foundationalist) can privately think about these statemetns/definitions as abreviating/meaning something different in their preferred conceptual vocabulary (or as meaning nothing in the case of the formalist), and no harm is done.

[discuss][Finally I should also flag that I am advocating a potentialist explication of talk about pure set theory within mathematics, and that this view does not directly imply anything about how to understand other kinds of mathematical talk. I suggest that in light of the burali forti paradox, claims about pure set theory are best clarified/cashed out in modal terms, as making claims about the 'logical possibility' of certain structures, rather than as quantifying over some collection of first order objects, as it initially appears to do. However, this does not imply that we should not be straightforward actualists about other mathematical structures which we do have a non-paradoxical categorical conception of, such as the natural numbers, or the complex numbers, of a hierarchy of sets with ur-elements which satisfies all the width requirements on a hierarchy of sets noted above and goes up exactly $\omega + \omega$.]

1.3 Outline

My plan of action will be as follows.

In chapter **??** I will argue that problems like the Buralli-Forti paradox motivate taking a potentialist approach to set theory. And then I will argue that potentialist set theory is best cashed out in terms of a certain notion of logical possibility and extendability. Independent considerations in the philosophy of logic motivate accepting a suitably free standing notion of logical possibility (which is not itself defined in terms of set theory), and that this notion usefully generalizes to a corresponding notion of logically possible extendability. And (I will argue that) employing this notion to formulate potentialist lets us illuminatingly simplify Putnam's and Hellman's formulations of potentialist set theory, and eliminate certain worrying conceptual problems/mysteries which beset Parsons' and Linnebo's formulations of potentialist set theory.

Now, if one provisionally accepts the potentialist interpretation of set theory is about advocated above, one faces a question about whether what mathematicians/set theorists are actually doing is justified (if they are so interpreted). Do all (or nearly all) the assumptins and inferences they make and find intuitive genuinely correspond to good arguments, if one explicates set theoretic notions in the way that I argue for? Presumably it's an important constraint on a good philosophical explaication of set theoretic language, that it not imply that contemporary set theorists are constantly getting things massively wrong, or making wildly unjustified inferences.

This brings us to the mathematical half of my project. Standard actualist conceptions of set theory actually have a little trouble vindicating mainstream matheamtical usage, insofar as there are some worries about how to justify use of all the ZFC axioms (particularly the axiom of replaceemnt). There is some apearance that we might have to settle for mere justification by fruitfulness not having found an problem yet etc. (a much weaker justification than we usually take ourselves have in mathematics) But I will show that, happily, adopting my prefered potentialist set theory lets us reconstruct all ZFC+FOL reasoning can be reconstructed using methods of inferences which are just as intuitively compelling and attractive. Thus the familiar hope that mathematical proofs can be justified on the basis of principles that seem prima facie obvious (if not completely indubitable) can be maintained if we accept the account of the nature of set theory which I propose.

Developing this kind of potentailist explication of set theory, as I proposed to do, may seem like rather a limited and obscure project, with little imapact on core questions in philosophy, or even core questions in the philosophy of mathematics. However, I will note at the end of chapter X that if my project succeeds, it has some interesting implications for larger questions. It motivates extreme truthvalue realism about all first order set theory (including thigns like GCH) and suggests an interesting line of attack on the access problem.

Chapter 2

Actualist Set Theory and Two Problems

Let me begin by introducing the potentialist approach to set theory and its motivation. I will then detail the two problems which the mathematical arguments in this monograph are intended to solve.

Potentialist approaches to set theory hold that when mathematicians make claims which appear to quantify over sets, we should understand them as claims about how it is possible to extend initial segments of the hierarchy of sets. More specifically, potentialists take set theorists to be making claims about how it would be possible to have objects which satisfy the non-height related requirements of ZFC, i.e., initial segments of the sets, and how it would be possible for such objects to be extended.

Potentialist approaches to set theory provide one popular and attractive response to Burali-Forti worries about the height of the hierarchy of sets, which arise as follows.

2.1 Actualist Set Theory and The Iterative Hierarchy Conception

2.2 A Problem Justifying Replacement

2.3 A Burlli-Forti/Arbitrariness Problem

Chapter 3

Potentialist Set Theory And Two Remaining Problems

There are well-known reasons for doubting that we have any coherent and adequate conception of absolute infinity (the supposed height of the hierarchy of sets). The concern here is not simply that it might be impossible to cash the notion of absolute infinity out in other terms. After all, every theory will have to take some notions as primitive. Rather, the worry is that it is logically impossible for any collection of objects to satisfy our intuitive notion of absolute infinity – just as Russell's paradox shows that it's logically impossible for any collection of objects to satisfy the axioms of naive set theory.

Our intuitive conception says that the hierarchy of sets goes all the way up – so no restrictive ideas of where it stops are needed to understand its behavior. However, if the sets really do go 'all the way up' in this sense, then it would seem that they should satisfy the following well-ordering principle.

> For any way some things could be well-ordered, there is an ordinal corresponding to it.

But the ordinals themselves are well ordered, and there is no ordinal corresponding to this well-ordering. Thus (it would seem), the naive well ordering principle above can't be correct.

The simplest response would seem to be to find some other restrictive characterization of the sets (in particular, some other characterization of the intended height of the hierarchy of sets).[1] However, it's not clear that any intuitive conception of the intended height of the sets remains once the paradoxical well-ordering principle above is retracted. As Wright and Shapiro put it [6], all our reasons for thinking that sets exist in the first place appear to suggest that, for any given height which an actual mathematical structure could have, the sets should continue up past this height.

Moreover, the sets lose a substantial aspect of their appeal as a mathematical foundation if we can't capture all talk of coherent mathematical structures within set theory – in the sense that all coherent mathematical structures have (something like) a model within the hierarchy of sets. However, it seems that this attractive principle will fail if the hierarchy of sets doesn't 'go all the way up' in the sense indicated above.

Informally, the axiom schema of replacement says that whenever some first order formula defines a function on a set $\boldsymbol{A}$, i.e., associates each element $\boldsymbol{x}$ of $\boldsymbol{A}$ with a unique $\boldsymbol{y}$, there is a set $\boldsymbol{B}$ equal to the image of $\boldsymbol{A}$.In other

[1] Note that the axioms of ZFC and even ZFC_2 don't suffice to categorically determine the height.

words the hierarchy of sets extends far enough up that all the elements in the image of A can be collected together.

Laying aside all questions about potentialism, there is a special question about how the axiom schema of replacement is justified. If we assume an actualist understanding of the axioms for set theory, them the truth of most of the ZF axioms seem to follow directly from the cumulative hierarchy conception of the sets. However (as Boolos famously emphasized [1]), unlike the other axioms, replacement seems to assert something about how high the universe of sets must extend which isn't obviously a consequence of our intuitive conception of the iterative hierarchy of sets.

One might think that the axiom of replacement could be justified by appeal to the intuitive idea that the hierarchy of sets goes 'all the way up' (one can always have a long well ordering which collects together initial segments witnessing all the relevant ϕ statements). But we have already seen that this idea leads to incoherence.

Instead, the axiom of replacement is often justified 'externally' by merely appealing to the fruitfulness of the consequences we can derive from it[2] (rather than deriving it from principles which themselves are immediately compelling as is the more usual practice in mathematics). I don't deny that such external justifications can provide some support. However, it would be appealing to have a more direct argument for a claim which we use as an unargued premise when reconstructing mathematical reasoning.

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[2] See Koellner on Godel on this [5]

3.1 Potentialist Set Theory as a Solution to Burali Forti

Potentialism, as developed by Putnam, Parsons and Hellman, provides a popular alternative approach to the above issue. Potentialistism holds that mathematical claims which appear to quantify over sets should (in some sense[3]) really be understood as claims about how it is logically possible to extend initial segments of the hierarchy of sets (i.e., collections of objects which satisfy our intuitive conception of the width of the hierarchy of sets but not the paradox-generating height requirement).

More specifically, potentialists take set theorists' singly-quantified existence claims, like $(\exists x)(x = x)$[4], to really be saying that that it would be logically possible for there to be an initial segment of the hierarchy of sets, V_0, containing an object x with the relevant property (here the property of

[3]Different potentialists may think of these explications of set theorists assertions in modal terms as either 'hermenutic' accounts of what contemporary set theorists already mean, or 'revolutionary' proposals for how our current mathematical concepts can be helpfully sharpened and modified in a neo-carnapian vein (to use Burgess and Rosen's terminology from []). I won't try to adjudicate this issue here, because it won't matter to the issues I will be discussing.

However, my preference is to advance potentialist paraphrases in a 'revolutionary' spirit, but only as a foundation for mathematics in the same way that Bourbaki-style reductions to set theory are currently employed as a foundations of mathematics. Thus, I'm not suggesting that set theorists should write proofs in my language, any more than Bourbaki were suggesting that number theorists should write out proofs in ZFC. Rather, I'm suggesting that we do set theory as usual, but officially note that we are (now) employing set theoretic statements as mere abbreviations for corresponding modal claims. Once we have vindicated the use of such abbreviations (by showing that all standard ZFC reasoning about set theory remains valid on the potentialist understanding as I do in this paper), set theorists can go on as usual without giving much thought to potentialism and logical possibility. However, we can pull out the fact that we are now employing set theoretic statements as abbreviations for corresponding modal claims when we need to answer philosophical questions such as the Burali-Forti problem discussed in this chapter.

[4]I mean instances of this claim as uttered in contexts where the realists would say that our quantifiers are implicitly restricted to the pure sets

being equal to itself). They take universal statements with a single quantifier like $(\forall x)(x = x)$, to really say that it is logically necessary that any object x in an initial segment of the hierarchy of sets would have the relevant property.

Potentialists handle nested quantification by using claims about how it would be logical possible for various initial segments of the hierarchy of sets to be extended. For example, they translate $(\forall x)(\exists y)(x \in y)$ as saying something like the following: 'necessarily, for any initial segment of the hierarchy of sets (call it V_1) and *any* set containing some set x, it is logically possible for there to be another initial segment (call it V_2) which extends[5] V_1 and contains a set y such that $x \in y$.

By adopting a potentialist understanding of set theory, we avoid commitment to arbitrary limits on the intended height of the hierarchy of sets. We also avoid the assumption that there is (or could be) any structure containing ordinals witnessing all possible well orderings. Nonetheless we make room for a sense in which the possibility of structures of arbitrary size can be relevant to the truth of set theoretic claims.

[Note that this may depend on rejecting putnams idea that there is also an actualist persepective on the hierarchy of sets which is somehow OK]

3.2 The Justification Problem Remains/Expands

In this paper/monograph I will attempt to address an important line of objection to potentialism. This objection concerns whether potentialists can make sense of current mainstream mathematical practice. It is not

[5]Meaning V_2 includes all the sets in V_1 and agrees with it regarding on the behavior of $\in$ within these sets.

immediately obvious that the ZFC axioms (especially the axiom of replacement) remain true when understood in a potentialist manner, as statements about the possible extendability of initial segments. Thus, it is not clear that the kind of arguments mathematicians actually produce still qualify as good arguments, once we accept a potentialist understanding of what the statements in these arguments mean. Accordingly, one might fear that accepting potentialism makes current mathematical practice look unjustified and count this as a reason to reject potentialist understandings of set theory. Indeed, it's not even clear whether first order logical derivations are still valid on a potentialist understanding as potentialist paraphrases of set theorists statements change their logical structure.[6]

Geoffrey Hellman (one of the most influential potentialists in the current literature) responds to the above problems by providing a kind of 'external' justification for the use of the ZFC axioms on (his version of the) potentialist approach to set theory. Hellman's justification goes like this. Assume that actualist set theory is true and there are cofinally many inaccessible cardinals. On this assumption, we can re-interpret (Hellman's preferred version of) potentialist claims as claims about what initial segments of the true hierarchy of sets exist. Then it is a theorem that, for each first order set theory sentence ϕ, this re-interpretation of the potentialist translation of ϕ will be true iff the original sentence ϕ is true. Thus, since ZFC is presumably true of the actualist hierarchy of sets, the potentialist translation of these claims

[6]Obviously first order derivations as applied to statements explicitly mentioning possible extendability are valid. The question is whether first order logic when applied to statements

will also come out true.[7]

However, (as Hellman himself explicitly notes) this justification is not satisfactory from a potentialist point of view, because it requires that we assume the existence of an actualist hierarchy of sets. Additionally, we must also assume that this hierarchy satisfies a further (somewhat) controversial large cardinal axiom: that there are co-finally many inaccessible cardinals.

Thus, if all we have is Hellman's story, it looks like adopting a potentialist understanding of set theory makes mathematicians' current proof practices look unjustified. For, the potentialist is moved by Burali-Forti worries to deny the existence (and even possibility) of an actualist hierarchy of sets, such that all possible 'initial segments of a hierarchy' (in the sense relevant to potentialism) could be thought of as initial segments of this single hierarchy. Thus they should not and cannot justify their foundational principles for reasoning in set theory by appeal to such a structure.

In this monograph, I aim to solve this problem by providing a more satisfactory potentialist justification for the use of ZFC in potentialist set theory: one which (unlike Hellman's external justification) does not depend on assumptions about the acceptability of actualist set theory or any other separate mathematical structure or practice.

[7] Hellman writes, "we may ask for comparisons between the [modal structuralist interpretation of set theory] and the usual fixed universe picture. It is not difficult to show that *from within the latter point of view* there is a compete agreement between the two with respect to all first order questions of ZF, decidable or not...the Putnam semantics [$\models_P$, which is interprets unbounded set theoretic claims as claims about what holds within certain initial segments of the hierarchy of sets V_k and how these V_k can be extended by other $V_{k'}$] gives answers to all mathematical questions. But does it give good ones? Yes, in this sense, it gives exactly the answers that the fixed set theoretic universe does, assuming the Axiom of Inaccessibles. That is we have a Correctness Theorem: Let $A(\mathbf{a})$ be a sentence of $\mathscr{L}(ZF^1)$ with parameters $\mathbf{a}(= a_1...a_n)$ and suppose that $V \models A(\mathbf{a})$, then $\exists k$ such that V_k is a full ZFC_2 model, the a_i are in its domain, and $V \models_P A(\mathbf{a})$."

Rather than translating sentences of potentialist set theory back into actualist set theory and then using ZFC to prove claims, I will articulate a formal system for reasoning about logical possibility. I will then show that that the potentialist translations of each of the ZFC axioms can be derived in this formal system, and that first order inferences can be safely made.[8]

Adopting my understanding of set theory also provides a new and interesting intrinsic justification for the axiom of replacement. Fortunately, it turns out that potentialists can provide just this kind of justification. I will show that the potentialist translation of the replacement schema can derived from principles which seem intrinsically plausible in their own right – not just externally attractive.

Thus, the long argument presented in this monograph will show that adopting potentialist approaches to set theory can help us solve *two* antecedent problems in the philosophy of set theory. In addition to providing a principled and elegant response to the Burali-Forti paradox, accepting potentialism also allow us to provide an appealingly intrinsic justification the axiom of replacement.

3.3 An Interpretation Problem Arises

What are these boxes being used, and what about the second order notion that is used to tie things together?

[8]I will also show that every first order logical deduction of a set theoretic sentence ϕ from premises Γ can be transformed into a valid deduction of $t(\phi)$ (the potentialist translation of ϕ) from $t[\Gamma]$ (the potentialist translations of all the sentences in the premise set Γ).

Chapter 4

The Language of Logical Possibility

Let me begin by introducing the concepts and notation I will use. Speaking generally, potentialist paraphrases of set theory make claims about how it would be logically possible to extend an initial segment of the hierarchy of sets.

Geoffrey Hellman's *Mathematics Without Numbers* [3] influentially formulated a version of potentialism using a logical possibility operator $\Diamond$, together with first and second order quantifiers which are allowed to reach inside the $\Diamond$ (so that we can say things like $\exists x \Diamond \phi$, and $\forall X \forall f \Diamond \psi$). However, it turns out to be possible to simplify this proposal. I will articulate a potentialist explication for set theory using only first order vocabulary and a single, fairly intuitive, notion of relativizable logical possibility (and not allowing quantifying in to the $\Diamond$).

Doing this will allow us to streamline our inference rules and sidestep the controversies about quantifying in discussed above.

4.1 Logical Possibility with Subscripts

Let me begin by precisifying the basic notion of logical possibility (denoted by $\Diamond$) at issue here. To evaluate whether a claim ϕ requires something logically possible (in this sense), we hold fixed the operation of logical vocabulary (like $\exists, \wedge, \vee, \neg$), but abstract away from any further metaphysically necessary constraints on the application of particular relation symbols. Thus, we consider all possible ways for relations to apply (including those ways that aren't definable). For example, it is logically possible that $(\exists x)(\text{Raven}(x) \wedge \text{Vegetable}(x))$, even if it would be metaphysically impossible for anything to be both a raven and a vegetable. We also abstract away from constraints on the size of the universe[1], so that $\Diamond(\exists x)(\exists y)(\neg x = y)$ would be true even if the actual universe contained only a single object. Note that this notion of logical possibility is not defined in terms of mere syntactic consistency within some formal deduction system.

Philosophers advocating a range of different philosophies of mathematics have invoked a similar notion.[2] This notion of logical possibility corresponds to our intuitive sense that certain descriptions of structures (like second order Peano Arithmetic[3]) require something coherent, while others

[1] See Etchemendy's [2] on the tension between standard Tarskian reinterpretation-based accounts of logical possibility and the intuitive notion of logical possibility regarding this point.

[2] maybe cite: Hartry Field, Shapiro, Rayo

[3] Note, however, that to assert a version of second-order Peano Arithmetic we will need to use relativized logical possibility, as we will see below.

(like Frege's inconsistent theory of extensions) do not.

I think we can also intuitively understand claims about logical possibility 'given' the facts about how certain relations apply. Consider a statement like the following.

> Given what cats and blankets there are, it is logically impossible that each cat slept on a different blanket last night.

This sentence has an intuitive reading which employs a notion of logical possibility *holding fixed the way that certain relations apply* (in this case, holding fixed what cats and blankets there are) rather than logical possibility *simpliciter*. A moment's thought will reveal that (on this reading) the above sentence is true if and only if there are more cats than blankets.

I propose to think of the logical possibility $\Diamond_{(...)}(...)$ as an operator which takes a sentence ϕ and a finite (potentially empty) list of relation symbols $R_1, ...R_n$ and produces a sentence $\Diamond_{R_1,...R_n}\phi$ which says that it is logically possible for ϕ to be true, without any change to how the relations $R_1, ...R_n$ apply. Thus, for example, the claim, 'Given what cats and baskets there are, it is logically impossible that each cat slept in a distinct basket' becomes:

$$\mathbf{C}\wedge\mathbf{B}{:}\neg\Diamond_{cat,basket}[(\forall x)(cat(x) \rightarrow (\exists y)(basket(y)\wedge sleptIn(x, y)\wedge(\forall z)[cat(z)\wedge sleptIn(z, y) \rightarrow x = z])]$$

Finally, note that by using this notion we can also make *nested* logical possibility claims, i.e., claims about the logical possibility of scenarios which are themselves described in terms of logical possibility. I have in mind sentences like the following:

$\Diamond$**C**$\wedge$**B**: $\Diamond(\neg\Diamond_{cat,basket}[(\forall x)(cat(x) \to (\exists y)(basket(y)\wedge sleptIn(x,y)\wedge$
$(\forall z)[cat(z) \wedge sleptIn(z,y) \to x = z])])$

The above sentence, $\Diamond$(C$\wedge$B), expresses a truth because (reading from the outside in):

- It is logically possible (holding fixed nothing) that there are 4 cats and 3 baskets.

- Relative to the logically possible scenario where there are 4 cats and 3 baskets, it is not logically possible (given what cats and baskets there are), that each cat slept in a basket and no two cats slept in the same basket.

Based on these kind of examples, I take logical possibility sentences of the form $\Diamond_{R_1 \dots R_n}\phi$ to be meaningful, even in cases where ϕ is itself a sentence which makes appeal to facts about logical possibility. As noted above, I will not allow sentences which quantify in to the $\Diamond$ of logical possibility.

To clearly express claims about logical possibility, we can define a formal language $\mathcal{L}$, which I will call the language of logical possibility (though no implication that this exhausts the concept should be drawn). Fix some infinite collection of variables and relation symbols of every arity together with $\perp$ and define $\mathcal{L}$ to be the smallest language built from these variables using these relation symbols and equality closed under applications of the normal first order connectives and quantifiers and $\Diamond_{\dots}$ (where $\Diamond_{\dots}$ expressions can only be applied to sentences (so there is no quantifying in). We will also use $\Box_{\dots}$ in our sentences but regard it as an abbreviation for $\neg\Diamond_{\dots}\neg$

4.2 Contrast With Other Modal Notions

Before going on, it may help philosophical readers to note how my notion of logical possibility differs from three vaguely similar modal notions in the literature (Tarskian re-interpretiability, metaphysical possibility and conceptual possibility) as follows.

The notion of logical possibility is (potentially) less demanding than the notion of Tarskian reinterpretability, for reasons discussed in Etchemendy's *The Concept Of Logical Consequence*. Essentially, the issue is that certain scenarios might be genuinely logically possible but require the existence of more objects than actually exist, and hence not permit any Tarskian reinterpretation (since Tarskian reinterpretations of a sentence must still take the sentence's quantifiers to range over some collection of objects in the actual world).

The notion of logical possibility is strictly less demanding than the notion of metaphysical possibility. For, as Frege noted, the laws of logic hold at all possible worlds. Yet (as noted above) statements like $(\exists x)Raven(x) \wedge Vegetable(x)$[4] can require something which is logically possible but metaphysically impossible.

Finally, the notion of logical possibility is also strictly less demanding than the notion(s) of idealized conceivability and/or conceptual possibility which occur in debates over philosophical zombies and Chalmers' *Constructing the World* (and are, inconveniently, sometimes also labeled logical possibility). For the notion of conceptual possibility reflects something like ideal

[4] I won't get into debates about what the true logical form of non-mathematical natural language sentences like 'something is both a raven and a vegetable' here.

a priori acceptability, so that when evaluating whether it is conceptually possible that ϕ we have to preserve all analytic truths associated with relations occurring in ϕ. In contrast (as I have noted above) logical possibility abstracts away from all such specific features of relations. Thus, for example, if it is analytic that $(\forall x)(bachelor(x) \rightarrow male(x))$, then it will be logically possible but *not* conceptually possible that $(\exists x)(bachelor(x) \wedge \neg male(x))$.

[Cashing set theory out in terms of logical rather than metaphysical possibility frees us from the presumption that there aren't limits on the number of physical objects (and perhaps the number of impure mathematical objects) generated by spacetime. Note that everything I say leaves it open for hardcore tractarains who reject this notion to say that logical possibility turns out to reflect the same thing as metaphysical possibility, just leaving room for different interpretations. I'm just noting that we have a concept of logical possibility independent of this, and that this suffices to give an attractive account of set theory.]

[okay somewhere you should define the notion of a formula taking a relation as a parameter...and you should be able to also make sure it lets one substitute in partially filled formulas, i.e. if $\phi(F)$ takes a two place relation F it makes sense to write $\phi(R(x, \cdot, \cdot))$ for a 3 place relation R.]

4.3 A Sample of this Language's Power: Describing Standard Models of the Natural Numbers

For reasons that will become clear in the next section, it will also be useful to categorically describe the intended structure of the natural numbers, using only my relativisable $\Diamond$ operator and other first order connectives.

One can uniquely describe the intended structure of the natural numbers by combining the first 6 Peano Axioms (which can be expressed using only first order logical vocabulary) with an Axiom of Induction, which can be expressed in the language of second order logic as follows:

$$(\forall X)(([0 \in X \wedge (\forall n)(n \in X \to S(n) \in X)] \to (\forall n)(\mathbb{N}(n) \to n \in X))$$ [5]

Informally, this axiom says that if some property applies to 0 and to the successor of every number it applies to, then it applies to all the numbers. We can express the same idea using $\Diamond$ $\Box$ (and any one place relation P other than '$\mathbb{N}$') as follows.[6]

$$\Box_{\mathbb{N},S}[P(0) \wedge (\forall x)(\forall y)(P(x) \wedge S(x,y) \to P(y))] \to (\forall x)(\mathbb{N}(x) \to P(x))$$

This formula says that, given the facts about what is a number and a successor, i.e., about how $\mathbb{N}$ and S apply), it would be logically impossible for P to apply to 0 and to the successor of each object which it applies to without applying to all the numbers.

Call the sentence you get by replacing the axiom of induction in second order Peano Arithmetic with the above modal sentence $PA_{\Diamond}$.

[5] Where 0 is not officially part of our langugage, but I use claims about 0 to abbreviate corresponding claims about the the unique number that isn't a successor of anything, in the usual fashion.

[6] Where $P(0)$ is shorthand for $(\exists z)(\forall w)(\mathbb{N}(z) \wedge \neg S(w,z) \wedge P(z))$.

Chapter 5

General Attractions of Cashing Out Mathematical Concepts in $\mathscr{L}$

5.1 A natural generalization of a notion that we already need

-We have separate reason for believing in logical possibility a la Field

-If you accept a notion of logical possibility, it seems only natural to be able to make sense of restricting that notion to the scenarios which preserve the structure of how some relation applies (in the actual world, or in some possible world under consideration).

5.2 Graspability in the Way McGee Suggests we Grasp the Natural Number Structure

-Expressing notions like PA2 in terms of logical possibility fits nicely with theories like mcGee's about how we can grasp the intended structure of the sets: we think that the current structure of the natural numbers makes it impossible for happiness or sadness or any property that we could introduce in language of chance physics could let us define appllies that way.

Chapter 6

Potentialist Paraphrases using $\mathscr{L}$

Let us now return to the subject of potentialist approaches to set theory. In this section, I will show how to use the notion of relitivizable logical possibility indicated above to provide attractive potentialist paraphrases for statements of (pure) set theory.

My potentialist paraphrases are inspired by Hellman's development of potentialism in *Mathematics Without Numbers*.[1]

[1] I mimic Hellman's story as far as possible. However, (as noted) where Hellman translates set theory by talking about the possibility of models of $\boldsymbol{ZFC}_2$, I do by talking about the possibility of standard-width initial segments – whatever their height I think this way of doing things is conceptually simpler and more elegant. Also (as noted above) it also lets us illuminate a way in which the axiom of replacement falls naturally out of the potentialist conception of set theory. (I also avoid Hellman's appeals to second order logic and quantifying in.)

6.1 Describing Standard-Width Initial Segments

Let me begin by introducing some definitions.

Recall the definition of well ordering from chapter9.1.1. I will define a formula $\mathcal{V}(set, ord, <, \in @\rangle)$ which intuitively says that some relation symbols $\langle$set, ord, $<$, $\in$ @$\rangle$ apply like the relations '...is a set', '...is an ordinal level of the hierarchy of sets' '...is an ordinal level of the hierarchy of sets below...' '...is an element of' and '...is a set which occurs by ordinal level...' would (respectively) apply within a standard initial segment of the hierarchy of sets.[2]

Definition 6.1.1. $\mathcal{V}(set, ord, <, \in @)$ is the conjunction of the following four requirements:

- The objects satisfying *ord* are well-ordered by $<$
- $(\forall x)(\forall y)[@(x, y) \rightarrow set(x) \wedge ord(y)]$
- Fatness: For each *ord* o, there are *set*s related to o by @ corresponding to all possible ways of choosing some of the *set*s which are available @ some ordinal $o' < o$ (in the sense of having exactly the chosen *set*s as elements).

[2]Thus we will, in effect, show how the notion of logical possibility can be used to specify what it takes for some relation symbols $set, \in$ etc. to apply as if to sets and ordinals within a *standard-width initial segments of the hierarchy of sets.*

This is no trivial task. Note that, for example, no sentence using only first order logical connectives can do it. All first order sentences describing the sets will have non-standard interpretations (indeed ones which are true of countable structures)

Philosophers of mathematics have traditionally tackled this problem by appealing to second order quantification to express the idea that each layer of sets must really contain objects corresponding to all possible subsets of the sets in lower layers. But we can express the same idea using the notion of relativizable logical possibility.

$$\Box_{set,ord,<,\in,@}(\forall o)[ord(o) \rightarrow$$

$$(\forall x)(P(x) \rightarrow set(x) \wedge (\exists o')(ord(o') \wedge o' < o \wedge @(x, o')))$$

$$\rightarrow$$

$$(\exists y)(set(y) \wedge @(y, o) \wedge (\forall z)(P(z) \leftrightarrow z \in y))]$$

[fix alignment][discuss]

Informally, this says that it would be impossible for a property P to apply to some sets related by @ to ords below o, without there being a set y such that $y@o$ which contains as elements exactly the sets which P applies to.

- Thinness: Only those sets guaranteed by fatness exist, i.e., ,
  - Every set is available at some ordinal level.

$$(\forall x)[set(x) \rightarrow (\exists o) ord(o) \wedge @(x, o)]$$

  - All sets available at some ord o can *only* have set elements which occur at some level below as elements.

$$(\forall x)(\forall o)(@(x, o) \rightarrow (\forall z)[z \in x \rightarrow \exists o' o' < o \wedge @(z, o')])$$

  - No two distinct sets have exactly the same set elements.

$$(\forall x)(\forall y)[set(x) \wedge set(y) \rightarrow x = y \vee (\exists z)(set(z) \wedge \neg(z \in x \leftrightarrow z \in y)]$$

- The $ords$ are disjoint from the sets

$$(\forall x)\neg(ord(x) \wedge set(x))$$

Note that this way of relating talk of sets to talk of ordinal levels differs slightly from the standard picture in that new sets occur at every level, whereas on the standard picture limit stages like ω just collect up the sets that occur at previous levels.

I will use $\mathcal{V}(V_i)$ to abbreviate the claim that $set_i, \in_i$ etc. satisfy the sentence $\mathcal{V}(set, ord, <, \in @\rangle)$ defined above.

6.2 The Translation

Recall that Potentialists propose to understand sentences of set theory by replacing apparent quantification over the sets with statements about how it would be possible to extend initial segments of the sets and choose elements from those initial segments, e.g., if ϕ is quantifier free then $\exists x\phi(x)$ would translate to $\Diamond[\mathcal{V}(\text{set}, \in ...) \wedge (\exists x)(\text{set}(x) \wedge \phi(x))]$ where this says that it would be logically possible for there to be an initial segment of the hierarchy of sets containing an object that satisfied ϕ.

To express potentialist truth conditions without quantifying in, I will require that each initial segment $\text{set}_i, \in_i, ord_i, <_i, @_i$ be paired with an associated assignment relation $\boldsymbol{R}_i$ which assigns each of the countably many variables x_1, x_2 (where the n-th successor of 0 stands in for x_n) in the first-order language of set theory to objects within set_i. When we ask about the possibility of extending the current initial segment $(\mathrm{set}_i, \in_i)$ we can relativize further $\Box$ and $\Diamond$ to $\boldsymbol{R}_i$ requiring that an extending model must have an $\boldsymbol{R}_{i+1}$ which must agree with $\boldsymbol{R}_i$ everywhere except for on the (number representing) the variable allowed to range over set_{i+1}.

Let us say that $\boldsymbol{R}$ represents a function from the objects satisfying $\boldsymbol{A}$ to those objects satisfying $\boldsymbol{B}$ if

- $\boldsymbol{R}$ is functional, i.e., $(\forall x)(\forall y)(\forall y')(\boldsymbol{R}(x, y) \wedge \boldsymbol{R}(x, y') \rightarrow y = y')$
- $\boldsymbol{R}$ maps from all of $\boldsymbol{A}$, i.e., $(\forall x)[\boldsymbol{A}(x) \rightarrow (\exists y)(\boldsymbol{R}(x, y))]$
- $\boldsymbol{R}$ maps to $\boldsymbol{B}$, i.e., $(\forall x)(\forall y)(\boldsymbol{R}(x, y) \rightarrow \boldsymbol{B}(y))$

I will use $\mathscr{V}(V_a)$ to abbreviate the claim that $\mathrm{set}_a, \in_a$ satisfy $\mathcal{V}(\mathrm{set}_a, \in_a, \boldsymbol{ord}_a, <_a, @_a)$ and $\boldsymbol{R}_a$ represents a function from the objects satisfying $\mathbb{N}$ to those satisfying set_a. More concretely, this amounts to the conjunction of the following three claims:

- $\mathcal{V}(V_a)$, i.e., $\mathrm{set}_a, \in_a$ ... behave like an initial segment of the hierarchy of sets.
- $\mathbb{N}, \boldsymbol{S}$ satisfy $\mathrm{PA}_\Diamond$ (the categorical description of the numbers above).
- $\boldsymbol{R}_a$ represents a function from the objects satisfying $\mathbb{N}$ to those satisfying set_a

Note that my only reason for using $\mathbb{N}$ is that the natural numbers (under successor) contain infinitely many definable objects, which we can use to represent variables, for example 1 represents x_1, 2 represents x_2 etc. In what follows, I will use $\mathbf{n}$, to abbreviate the formula where $\mathbf{n}$ is replaced by a variable constrained to be the (unique) n-th successor of 0. I will use subscripts of the form $\Diamond_{V_n}$ and $\Box_{V_n}$ to abbreviate claims of the form $\Diamond_{\text{set}_n,\in_n,ord_n,@_n,\leq_n,\mathbb{N},S,R_n}$ and $\Box_{\text{set}_n,\in_n,ord_n,@_n,\leq_n,\mathbb{N},S,R_n}$.

I will use $V_a \geq V_b$ to abbreviate the claim that the set_a, ord_a under $\in_a, @_a, \leq_a$ extends the set_b ord_b under $\in_b, @_a, <_a$.

- $\mathcal{V}(V_a)$
- $\mathcal{V}(V_b)$
- $(\forall x)[\text{set}_b(x) \to \text{set}_a(x)]$
- $(\forall x)(\forall y)[\text{set}_b(y) \to (x \in_b y \leftrightarrow x \in_a y)]$
- $(\forall x)[\text{ord}_b(x) \to \text{ord}_a(x)]$
- $(\forall x)(\forall y)[\text{ord}_b(y) \to (x <_b y \leftrightarrow x <_a y)]$
- $(\forall x)(\forall y)[\text{ord}_b(y) \to (x @_b y \leftrightarrow x @_a y)]$

I will use $\vec{V}_a \geq_{\mathbf{x}} \vec{V}_b$ to abbreviate the claim that $V_a \geq V_b$ and the assignment of variables R_b agrees with R_a everywhere *except on* $\mathbf{x}$. Put more concretely, this is to say that

- $\mathbb{N}, S$ satisfy $\text{PA}_\Diamond$.
- R_a represents a function from the objects satisfying $\mathbb{N}$ to those satisfying set_a
- R_b represents a function from the objects satisfying $\mathbb{N}$ to those satisfying set_b

- $(\forall n)[\mathbb{N}(n) \to n = \mathbf{x} \vee (\forall y)(\boldsymbol{R}_a(n, y) \leftrightarrow \boldsymbol{R}_b(n, y))]$

We can now translate the set theoretic utterance $(\exists x)(\forall y)[x = y \vee \neg y \in x]$ into a potentialist claim about how it is logically possible for $\text{set}_1, \in_1, \boldsymbol{R}_1$ to be extended. First we rewrite this set theoretic statement in a regimented language with numbered variables as $(\exists x_1)(\forall x_2)[x_1 = x_2 \vee \neg x_2 \in x_1]$. Then we translate this sentence into:

$$\Diamond(\mathscr{V}(V_1) \wedge \Box_{V_1}[\vec{V}_2 \geq_2 \vec{V}_1 \to (\forall z)(\forall y)(\boldsymbol{R}_2(\mathbf{1}, z) \wedge \boldsymbol{R}_2(\mathbf{2}, y) \to z = y \vee \neg y \in_2 z)])$$

In words, such $\exists x_2 \forall x_1$ sentences can be understood as making a claim with, essentially, the following form. There could be a model of set theory $\text{set}_1, \in_1$ [more pedantically: a model satisfying the *width* requirements of set theory] and a relation $\boldsymbol{R}_1$ assigning 1 (representing x_1) to an element of set_1 so that it is necessary (holding fixed $\text{set}_1, \in_1, \boldsymbol{R}_1$ and the numbers) than any model of set theory $\text{set}_2, \in_2$ extending $\text{set}_1, \in_1$ and relation $\boldsymbol{R}_2$ assigning 2 to an element of set_2 (while agreeing with $\boldsymbol{R}_1$ about the assignment of 1) makes the interior of the above formula true when x_1, x_2 are replaced by the assignments of $1, 2$ by $\boldsymbol{R}_2$ and $\in$ is replaced with $\in_2$.

We will make one small change to the strategy illustrated above to allow us to the quantifiers in a uniform fashion. In the above examples the first quantifier had to be treated in a special manner as (the relations abbreviated by) V_1 were not required to 'extend' any V_0. To this end, our translations will introduce a V_0 and insist that $V_1 \geq V_0$. Thus, for example, my official translation of $(\exists x)(\forall y)[x = y \vee \neg y \in x]$ is actually:

$$\Diamond[\mathscr{V}(V_0) \wedge \Diamond(\vec{V}_1 \geq_2 \vec{V}_0 \wedge \Box_{V_1}[\vec{V}_2 \geq_2 \vec{V}_1 \rightarrow (\forall z)(\forall y)(R_2(\mathbf{1}, z) \wedge R_2(\mathbf{2}, y) \rightarrow z = y \vee \neg y \in_2 z)])]$$

I will now describe recursive principles which let us translate every sentence in the first-order language of set theory into a claim about logically possible extendability.

First we define a partial paraphrase function t_n, as I do below. Intuitively, $t_n(\phi)$ transforms a set theoretic formula ϕ into the a potentialst claim about how the initial segment V_n and assignment function f_n (coded by our assignment relation R_n) can be extended so as to satisfy (a potentialist version of) ϕ – while holding fixed f_n's current assignments to all numbers representing variables which occur free in ϕ.

Definition 6.2.1. For any number n and set theoretic formula ϕ...

- $t_n(x_i \in x_j)$ is the claim that R_n assigns the godel number of x to an object $\in_n$ the object it assigns to the godel number of y,i.e.,$(\forall z)(\forall z')[R_n(\mathbf{i}, z) \wedge R_n(\mathbf{j}, z) \rightarrow z \in_n z']$

- $t_n(x_i = x_j)$ is the claim that R_n assigns i to the same object it assigns j to i.e.,$(\forall z)(\forall z')[R_n(\mathbf{i}, z) \wedge R_n(\mathbf{j}, z') \rightarrow z = z']$

- $t_n(\neg\phi) = \neg t_n(\phi)$

- $t_n(\phi \vee \psi) = t_n(\phi) \vee t_n(\psi)$

- $t_n((\forall x)\phi(x))$ is the claim that $\Box_{V_n}[V_{n+1} \geq_{\mathbf{x}} V_n \to t_{n+1}(\phi)]$, where $\Box_{V_n}/\Diamond_{V_n}$ abbreviates a claim about what is logically necessary/possible holding fixed the facts about $\text{set}_n, \in_n, \boldsymbol{ord}_n, @_n, \leq_n, \mathbb{N}, \boldsymbol{S}, \boldsymbol{R}_n$.

- $t_n((\exists x)\phi(x))$ is the claim that $\Diamond_{V_n}[V_{n+1} \geq_{\mathbf{x}} V_n \wedge t_{n+1}(\phi)]$

The translation of a set theoretic sentence ϕ is $t(\phi) = \Box[\mathcal{V}(V_0) \to t_0(\phi))]$.

Note that the validity of the above translation relies on the fact that for any two structures satisfying ZFC_2 one is isomorphic to an initial segment of the other. Also note that in the above definition we can replace $\boldsymbol{V}_j$ with $V_{j \bmod 2}$ without affecting the truth value of the translation. This allows us to translate sentences with arbitrarily many quantifier alternations using a fixed finite number of atomic relations.

[discuss: new][In what follows, I will sometimes use $\phi(f_n(x_i))$ to abbreviate claims of the form $(\exists k)\boldsymbol{R}_n(\mathbf{i}, k) \wedge \phi(k)$, and f_n to abbreviate the list of relations $\boldsymbol{R}_n, \mathbb{N}, \boldsymbol{S}$. For ease of reading, I will also sometimes use variables $x, y, z...$ rather than $x_0, x_1, x_2....$]

6.3 Note about these translations

Lemma 6.3.1. *If $\phi, \theta_1, \dots, \theta_n$ are formula in the language of set theory then*

1. $t_n(\phi)$ *is always content-restricted to* $\boldsymbol{V}_n, \boldsymbol{R}_n, \mathbb{N}, \boldsymbol{S}$

2. *If ϕ is a sentence, then $t(\phi)$ is content restricted to the empty list.*

3. *For all i, j if $\mathscr{V}(V_i), t_i(\theta_1), \dots, t_i(\theta_n) \vdash_{\Diamond} t_i(\phi)$ then $\mathscr{V}(V_j), t_j(\theta_1), \dots, t_j(\theta_n) \vdash_{\Diamond} t_j(\phi)$*

Proof. Claims 1 and 2 follow immediately from the definition. Claim 3 follows by a tedious, but simple, induction on proof length, where we transform the t_i version of a proof to the t_j version by replacing every instance of a relation in V_{i+k}, f_{i+k} with the corresponding relation V_{j+k}, f_{j+k} and noting that the result is still a proof. □

6.4 Advantages over existing versions of Putam-Hellman

-Linnebo suggests that Hellman faces problems about metaphysical shyness, suggesting that to make sense of set theory we will want to say two possible structures could be put together (and indeed my principle of pasting says just that), yet . But my way of doing things in terms of structural extendability doesn't make any claim about the essences of objects, just that certain structures could be combined.

-We also Helman's quantifying in hence disagreement about what to say is true of an object in logically possible scenarios where it doesn't exist, and can continue to use classical first order logic within consideration of each logically possible scenario.

-And we nudge off Hellmans worries about second order logic, by eliminating it in favor of a notion that he essentially already has to accept to make sense of applied math (how the structure of some objects makes it possible for these objects to be extended). Hellman considers using meriology in place of second order logic, but this would invovle taking the principles of mereology to be logically necessary.

-It can also be noted that my way of doign things drops hellmans strong conditions on what Vs must be like, in favor of somethign closer to the iterative heirarchy conception boolos introduces. This makes it easier to justify basic principles, since the truth of pairing etc. doesn't depend on the logical possibility of there being something which satisfyies second order replacement.

-A cost of my strategy is that it doesn't let us say what hellman does about second order quantification over the sets. But this isnt much of a cost, since what hellman says isn't very attractive, e.g., as he himself admits second order replacement doesn't justify first order replacement. Also I think the core potentialist idea kindof suggests that second order quantification shouldn't make sense. There isn't some logically possible first order collection, the sets, such that we can ask what subsets of it exist.

-I suspect that hellman himself only uses this strategy because it lets us give a justification of potentialist set theory from an actualist point of view, via X theorem. But of course my aim in this book will be to justify potentialism directly.

6.5 Advantages over Parsons-Linnebo

Putnam and Hellman represent one classic and influential way of thinking about set theory. But Parsons and Linnebo represent a different tradition, which is also worth considering. While Putnam and Hellman take the modal notion involved in potenatialist set theory to reflect general constraints on what's logically possible for all objects, Linnebo develops some ideas from

Charles Parsons to motivate a version of potentialist set theory which renounces such general logicist ambitions.

-Linnebo notes that parsons is unclear about what modal notion to use, and suggests that we use two notions, metaphysical possibility and interpreational possibility. Here the idea is that we avoid the problem of metaphysical shyess by saying that set theory explores special objects, sets, whose essences are known to allow free recombination. Rather than appealing to general laws of logical possibility, we appeal to metaphysical possibility and an "interpretational possibility" which characterizes (in some sense) what sets one could take there to be in a given metaphysically possible world.

He takes the iterative heirarchy conception of sets to reflect constraints on what sets there are, so all acceptable interpreations of set theory must satisfy this.

[inc or? He invokes a kind of grounding intuition as relevant to this, saying that that the existence of sets must be grounded in the existence of their elements, connecting the iterative heirarchy of conception of sets (which for all I'm concerned might just be the culturally arbitrary first attractive solution[i.e. precivification of the naive concept of sets] we happened to hit on to russells paradox) to a kind of grounding intuition. This idea of arguing that the iterative hierarchy conception is the most metaphysically natural successor to the naive concept of sets is interesting, but I don't think we should make the truth of basic claims in set theory depend on it. Linnebo and I are both happy to give mathematicians (and ordinary speakers) significant choice of which logically coherent mathematical structures to talk in terms of outside of set theory, so I don't think there's much motivation for arguing that potentialist set theory with the iterative hierarhcy conception is the unique metaphysically natural choice for udnerstanding set talk,when it comes to that..]

Furthermore, to avoid the hastles about quantifying over objects in scenarios where they don't exist, Linnebo takes this intepreational possibility not to satisfy S5, but to have an accessibility relation which only lets one scenario be possible with respect to another if the first is strictly larger.

-In explicitly saying that set theory involves reflection on a specific kind of thing, the sets, and a notion of interpretational possibility (which will see rather sui generis notion of interpretational possibility), Linnebo drops much of the hope of helping with the access problem by tying knowledge of pure set theory to laws of logic which apply to everything (and hence might be learnable by dealing with less refined hard to meet objects than the sets)

-But this concept of interpretational possibility can't just be interpretation in the familar tarsiakian sense, which makes one choose from within a fixed universe of objects existing at some metaphysically possible world. Otherwise the arbitrariness which potentialism promised to let us avoid gets dragged back in. Instead I think he must mean interpreational possibility in a quantifeir variantist sense, where you get to choose between arbitrary logically possible structures for candidate disambiguations of what sets there are . So I think his notion of interpreational possibility needs to build in something very much like my notion of logical possibility.

-Furthermore, this notion of interpretational possibility can't work like normal appeals to acceptable interpretations. For normally wheren there are many acceptable interperations of how mountain or martinia apply we say that it is vague whether a given borderline case is a martini. But we certainly don't say that its *vague* whether there are e.g. sets above Vomega. Rather (hellman and linnebo and parsons and I all agree) we say that this is true because the modal translation is true. So I think the parsons/hellman approach does a better job of relating the formal stuff we want to say about potentailist set theory to antecedently familiar concepts.

-Linnebo also avoids the hastle of having to use free logic by considering only possible scenarios that extend one another, but this takes his modal notion further away from both logical possibility and interpreational possibility, unless you imagine some person constructing the sets/commiting to interepting the concept "set" more and more demandingly in time. Presumably Linnebo doesnt think that we perform such constructions in time (and its not clear that time even could have arbitrary cardinalities, or what such acts of construction would look like) so its deeply unclear what this is supposed to mean. Are we talking about it being metaphysically possible to say let there be layer and then another layer etc. Ambiguity about this key philosophical question really stands out in linnebo and shapiro's work on potentialism, so I think banishing it is a serious advantage.

[I won't consider the question of whether linnebo's system does a better job of capturing how intuitionists could think about the sets, because I've never found any satisfying arguments for intuitionsim, and i'm trying to provide an intrinsically attractive foundation for set theory which dispells confusions, not model all vague and potentailly ill definde philosophical ideas people have ever had about set theory.]

Chapter 7

The Formal System I: Basic Rules

7.1 Rules Inherited from First Order Logic

With the language of logical possibility $\mathcal{L}$ in place, I will now introduce some inference rules for reasoning about logical possibility. I will recursively define the set of strings which constitute proofs in my deductive system by listing closure conditions in this chapter and the next.[1] Let $\Gamma, \Gamma_1, \Gamma_2$ be finite sets of formulas, and $\Gamma \vdash \theta$ express the claim that one can prove θ given the assumptions in Γ.

My closure conditions begin, boringly, with the following principles cor-

[1] As usual, I will say that all variables that occur in an atomic formula are free. If a variable occurs free (or bound) in θ or in ψ, then that same occurrence is free (or bound) in $\neg\theta$, $(\theta \wedge \psi)$, $(\theta \vee \psi)$, and $(\theta \rightarrow \psi)$ and $\Diamond\theta$ and $\Box\theta$. That is, the (unary and binary) connectives do not change the status of variables that occur in them. All occurrences of the variable v in θ are bound in $\forall v\theta$ and $\exists v\theta$. Any free occurrences of v in θ are bound by the initial quantifier. All other variables that occur in θ are free or bound in $\forall v\theta$ and $\exists v\theta$, as they are in θ.

responding to standard inference rules for first order logic,(which I take from the Stanford Encyclopedia article on classical logic[2]).

(As) If ϕ is a member of Γ, then $\Gamma \vdash \phi$.

($\wedge$I) If $\Gamma_1 \vdash \theta$ and $\Gamma_2 \vdash \psi$, then $\Gamma_1, \Gamma_2 \vdash (\theta \wedge \psi)$.

($\wedge$E) If $\Gamma \vdash (\theta \wedge \psi)$ then $\Gamma \vdash \theta$; and if $\Gamma \vdash (\theta \wedge \psi)$ then $\Gamma \vdash \psi$.

($\vee$I) If $\Gamma \vdash \theta$ then $\Gamma_1 \vdash \theta \vee \psi$; if $\Gamma \vdash \psi$ then $\Gamma \vdash \theta \vee \psi$.

($\vee$E) If $\Gamma_1 \vdash (\theta \vee \psi)$, $\Gamma_2, \theta \vdash \phi$ and $\Gamma_3, \psi \vdash \phi$, then $\Gamma_1, \Gamma_2, \Gamma_3 \vdash \phi$.

($\rightarrow$I) If $\Gamma, \theta \vdash \psi$, then $\Gamma \vdash (\theta \rightarrow \psi)$.

($\rightarrow$E) If $\Gamma_1 \vdash (\theta \rightarrow \psi)$ and $\Gamma_2 \vdash \theta$, then $\Gamma_1, \Gamma_2 \vdash \psi$.

($\neg$I) If $\Gamma_1, \theta \vdash \psi$ and $\Gamma_2, \theta \vdash \neg\psi$, then $\Gamma_1, \Gamma_2 \vdash \neg\theta$.

(DNE) If $\Gamma \vdash \neg\neg\theta$ then $\Gamma \vdash \theta$.

($\forall$E) If $\Gamma \vdash \forall v \theta$, then $\Gamma \vdash \theta(v|v')$, provided that v' is free for v in θ.[3]

($\forall$I) If $\Gamma \vdash \theta$ and the variable v does not occur free in any member of Γ, then $\Gamma \vdash \forall v \theta$.

(=I) $\Gamma \vdash v = v$, where v is any variable.

(=E) If $\Gamma_1 \vdash v_1 = v_2$ and $\Gamma_2 \vdash \theta$, then $\Gamma_1, \Gamma_2 \vdash \theta'$, where θ' is obtained from θ by replacing zero or more occurrences of v_1 with v_2, provided that no bound variables are replaced, and all substituted occurrences of v_2 are free.

($\bot$ I) If $\Gamma \vdash \psi \wedge \neg\psi$ then $\Gamma \vdash \bot$.

($\bot$ E) If $\Gamma, \theta \vdash \bot$ then $\Gamma \vdash \neg\theta$.

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[2] http://plato.stanford.edu/entries/logic-classical/ , with straightforward simplifications arising from the fact that my language $\mathcal{L}$ does not contain any constants

[3] That is, if substituting v with v' does not lead to any variable which was antecedently free becoming bound. Here $\theta(v|v')$ stands for the result of substituting *all* free instances of v in θ with instances of v'.

For convenience, I will also include the following inference rules for $\exists$ whose validity is straightforward to demonstrate using the definition of $\exists$ (as an abbreviation for $\neg\forall\neg$).

($\exists$I) If $\Gamma \vdash \theta$, then $\Gamma \vdash \exists v\theta'$, where θ' is obtained from θ by substituting the variable v' for zero or more occurrences of a variable v, provided that (1) all of the replaced occurrences of v are free in θ, and (2) all of the substituted occurrences of v' are free in θ'.

($\exists$E) If $\Gamma_1 \vdash \exists v\theta$ and $\Gamma_2, \theta \vdash \phi$, then $\Gamma_1, \Gamma_2 \vdash \phi$, provided that v does not occur free in ϕ, nor in any member of Γ_2.

In order to state analogous inference rules for the $\Box$ and $\Diamond$, I will define a sense in which a sentence can be *content restricted* to a finite list of relations $\mathcal{L}$. Note that, just like the relations subscripted by a $\Diamond$ or $\Box$, the order of the relations in $\mathcal{L}$ does not matter, so we may freely take intersections or talk of one list containing another without concern for order.

7.2 Basic $\Diamond$ Rules

7.2.1 Content-restriction

In reasoning about logical possibility, it will be useful to distinguish a class of sentences whose truth depends only on the facts about a list $\mathcal{L}$ of relations, i.e., those sentences ϕ such that $\Diamond_{\mathcal{L}}\phi$ intuitively entails ϕ. We will call such a sentence *content restricted to* $\mathcal{L}$. For example, if $\mathcal{L}$ is the list $\text{Person}(\cdot), \text{Loves}(\cdot, \cdot)$ then the claim 'every person loves something', i.e., $(\forall x)\,[\text{Person}(x) \implies (\exists y)\text{Loves}(x, y)]$, is content restricted to $\mathcal{L}$. In contrast the sentence 'every thing loves some thing', i.e., $(\forall x)(\exists y)(\text{Loves}(x, y))$, is not content restricted as it's truth depends on the existence of objects that neither Person nor Loves[4] applies to. As these examples suggest, sentences are content restricted if only the relations from $\mathcal{L}$ are mentioned and every quantifier is restricted to range over elements that belong to some tuple in the extension of a relation in $\mathcal{L}$. The following definitions capture this intuition.

Definition 7.2.1. Let $y \in \mathrm{Ext}(R_1, \dots R_n)$ abbreviate the formula

$$\bigvee_{\substack{1 \le i \le n \\ 1 \le j \le l_i}} (\exists x_1) \dots, (\exists x_{j-1}), (\exists x_{j+1}), \dots, (\exists x_{l_i}) R_i(x_1, \dots, x_{j-1}, y, x_{j+1}, \dots, x_{l_i})$$

where l_i is the arity of R_i and $\bigvee_{\substack{1 \le i \le n \\ 1 \le j \le l_i}} \phi_{i,j}$ indicates the disjunction $\phi_{i,j}$ over all indicated values for i and j.

Thus, $y \in \mathrm{Ext}(R_1, \dots R_n)$ is the formula asserting that some tuple $\vec{v}$ including y satisfies some $R_i(\vec{v})$.

Definition 7.2.2. I will say that a sentence ϕ is **explicitly content-restricted** to a list $\mathcal{L}$ if it is a member of the smallest set S satisfying:

- $\perp$ is in S
- If v_i, v_j are variables the formula $v_i = v_j$ is in S
- If v_i is a variable and $R_i \in \mathcal{L}$ then $R_i(v_j)$ is in S
- If $\psi \in S$ and $\rho \in S$ then $\neg\psi$, $\psi \vee \rho$, $\psi \wedge \rho$ and $\psi \rightarrow \rho$ are all in S

[4]By this, I mean objects which are not part of any pair in the extension of Loves.

- If $\psi \in S$ and $\mathcal{L}$ is non-empty then $\exists y(y \in \text{Ext}(\mathcal{L}) \wedge \psi)$ is in S
- If $\psi \in S$ and $\mathcal{L}$ is non-empty then $\forall y(y \in \text{Ext}(\mathcal{L}) \rightarrow \psi)$ is in S
- If $\phi = \Diamond_{\mathcal{L}'}\psi$, where ψ is a sentence and $\mathcal{L}' \subseteq \mathcal{L}$ then $\phi \in S$. Note that ψ *need not be in* S

The last clause is motivated by the fact that the truthvalue of $\Diamond_{\mathcal{L}'}\psi$ is completely determined by facts about the relations in $\mathcal{L}'$. Furthermore, as no free variables are allowed in $\Diamond_{\mathcal{L}'}\psi$ its truth value is unaffected by any external quantification.

Thus, for example, if $\mathcal{L}$ is a list that contains (exactly) a two-place relation R and a one place relation Q, then $(\forall x)(\forall y)(x = y)$ is not content-restricted to $\mathcal{L}$. Neither is $(\exists x)(Q(x) \wedge K(x))$. But $(\forall x)[x \in \text{Ext}(R) \rightarrow (\forall y)(y \in \text{Ext}(R) \rightarrow [R(x, y) \rightarrow Q(y)]])$[5] (which is first order logically equivalent to $(\forall x)(\forall y)[R(x, y) \rightarrow Q(y)])$ is content-restricted to $\mathcal{L}$. And so is $\Diamond_R[(\forall x)(R(x, x) \wedge (\exists y)S(x, y))]$.

Also note the following consequences of the definition above:

- If $\mathcal{L}$ is a sublist of $\mathcal{L}'$, then all formulae ϕ which are content restricted to $\mathcal{L}$ are also content restricted to $\mathcal{L}'$.
- A sentence is content restricted to the empty list $\mathcal{E}$ iff it is a truth functional combination of unsubscripted $\Box$ or $\Diamond$ sentences or $\bot$.

As you may have noticed, explicitly content-restricted sentences are generally long and unwieldy. This can be annoying when writing up proofs

[5] i.e. $(\forall x)[(\exists k)(R(x, k) \vee R(k, x)) \rightarrow (\forall y)[(\exists k')(R(y, k') \vee R(k', y)) \rightarrow (R(x, y) \rightarrow Q(x))]]$

whose inference steps can only (strictly speaking) be applied to sentences which are content-restricted to some list $\mathcal{L}$. To avoid this annoyance, I make the following definition.

Definition 7.2.3. A sentence ϕ is **implicitly content-restricted** to $\mathcal{L}$ if there is a sentence ψ explicitly content restricted to $\mathcal{L}$ and $\phi \leftrightarrow \psi$ can be derived (using no assumptions) using the above inference rules.

I will then frequently use the short hand of applying rules which (strictly speaking) can only be applied to content-restricted sentences to implicitly content restricted sentences – taking the work of using first order logic to deduce the explicitly content-restricted form of a sentence before applying the relevant rule (and then transforming it back after applying the rule) for granted.

7.2.2 Rules

I can now introduce the core inference rules and axiom schemas which govern reasoning with $\Box$ and $\Diamond$ in my formal system.

($\Diamond$ I) Diamond Introduction. If $\Gamma \vdash \theta$ and θ is a sentence, then $\Gamma \vdash \Diamond_{\mathcal{L}}\theta$

This rule captures the idea that what is actual must also be logically possible, even while holding fixed the facts any list of relations $\mathcal{L}$ one might care to specify.

Examples:

- "There are two cats" $\Rightarrow$ "It is logically possible, given what cats there are, that there are two cats".

- "There are two cats" $\Rightarrow$ "It is logically possible, given what dogs there are, that there are two cats".

($\Diamond$ E) Diamond Elimination. If $\Gamma \vdash \Diamond_L \theta$ and θ is content-restricted to $\mathcal{L}$ then $\Gamma \vdash \theta$

This rule expresses the idea that when θ is content-restricted to $\mathcal{L}$, the truth value of θ is totally determined by the facts about $\mathcal{L}$.

For instance:

- "It is logically possible, given what cats there are, that there are two cats" $\Rightarrow$ "There are two cats"
- BUT NOT: "It is logically possible, given what dogs there are, that there are two cats" $\Rightarrow$ "There are two cats"

Note that the second inference is not permitted by my rule because θ ("there are two cats") is not content-restricted to the list $\{dog(\cdot)\}$

(In$\Diamond$) Inner Diamond. [u sure u don't want to write this in the way that K is written in S5]

Suppose $\Gamma_1 \vdash \Diamond_L \theta$. If $\Gamma_2, \theta \vdash \phi$, where $\Gamma_2 = \gamma_1 ... \gamma_m$ is a list of sentences which are content-restricted to $\mathcal{L}$. Then $\Gamma_1, \Gamma_2 \vdash \Diamond_{\mathcal{L}'} \phi$ for any $\mathcal{L}' \subseteq \mathcal{L}$.

This inference rule captures reasoning of the following form. Given the facts about $\mathcal{L}$, it's possible that θ. Any scenario where θ is true while the facts about $\mathcal{L}$ are held fixed must also be one in which the premises $\gamma_1 \wedge ... \wedge \gamma_m$ are true (because these sentences are content-restricted to $\mathcal{L}$). Thus it must be possible, given the facts about $\mathcal{L}$, that $\theta \wedge \gamma_1 \wedge ... \wedge \gamma_m$. As a matter of logic, any scenario in which $\theta \wedge \gamma_1 \wedge ... \wedge \gamma_m$ is one in which ϕ. So, it must be possible given the facts about $\mathcal{L}$ that ϕ. And since $\mathcal{L}' \subseteq \mathcal{L}$, ϕ must also but possible holding fixed only the facts about $\mathcal{L}'$.

I will use some visually suggestive notation to keep track of inferences of this form, as follows:

1	γ	Assump [1]
2	$\Diamond_{\mathcal{L}}\theta$	Assump [2]
3	$\Diamond$ θ $[\mathcal{L}]$	2, In$\Diamond$I [2]
4	γ	1, import [1]
5	...	
6	ϕ	[1,2]
7	$\Diamond_{L}\phi$	2,3-6 In$\Diamond$E [1,2]

Intuitively speaking, the forked line going from 3-6 above separates off a location for reasoning about a logically possible scenario in which θ is true while all the facts about $\mathcal{L}$ in our current context are preserved.

A line ρ can be written down inside the "$\Diamond_{\mathcal{L}}$ context" governed by the claim that $\Diamond_{\mathcal{L}}\theta$ if

- $\rho = \theta$
- $\rho = \gamma$ for some γ which is content-restricted to $\mathcal{L}$ and occurs on an ear-

lier line in the proof which is in the same context as the $\Diamond_{\mathcal{L}}\theta$ statement used to introduce this inner diamond context.

- ρ follows from previous lines within this $\Diamond$ context by one of the inference rules for reasoning about logical possibility presented in this paper.

One can leave $\Diamond_{\mathcal{L}}$ context above by going from knowledge that ϕ holds within this context to the conclusion that $\Diamond_{\mathcal{L}'}\phi$ holds outside it, provided that $\mathcal{L}'$ is a sublist of $\mathcal{L}$.

Example of Deploying In$\Diamond$:

Consider the following very short argument.

> Given what cats and hunters there are, its logically possible that something is both a cat and a hunter.⇒ Given what cats there are, its logically possible that something is both a cat and a hunter.

We can capture this argument in my system as follows.

1	$\Diamond_{cat,hunter}(\exists x)(cat(x) \land hunter(y))$	[1]
2	$\Diamond$ $(\exists x)(cat(x) \land hunter(y))$ $[cat, hunter]$	1 In$\Diamond$I, [1,2]
3	$(\exists x)(cat(x) \land hunter(y))$	2 repetition[6] [1,2]
4	$\Diamond_{cat}(\exists x)(cat(x) \land hunter(y))$	2-3, In$\Diamond$E [1]

[6] strictly speaking this repetition is not necessary

Thus $\Diamond_{cat,hunter}(\exists x)(cat(x) \wedge hunter(y)) \vdash \Diamond_{cat}(\exists x)(cat(x) \wedge hunter(y))$. Note that $\{cat\}$ is a sublist of $\{cat, hunter\}$ and no extra premises are used in the deduction of ϕ from ϕ, so the requirements for Inn$\Diamond$E are satisfied.

Note: the requirement that each γ_i be content-restricted to $\mathcal{L}$ prevents us from importing facts about objects which don't satisfy any of the relations in $\mathcal{L}$ into our reasoning about what the relevant scenario $\mathcal{L}$ must be like. For example, consider the following *invalid* inference.

> "There is something that is not a cat." "It is logically possible, given what cats there are, that everything is a cat." $\Rightarrow$ "It is logically possible given what cats there are, that everything is a cat and something is not a cat."

1	$(\exists x)\neg cat(x)$	[1]
2	$\Diamond_{cat}(\forall y)cat(y)$	[2]
3	$\Diamond$ | $(\forall y)cat(y)$ $[cat]$	2, In$\Diamond$I
4	| $(\exists x)\neg cat(x)$	2 import [1] (**INVALID**)
5	| $(\exists x)\neg cat(x) \wedge (\forall y)cat(y)$	3,4 $\wedge$ I [1,2,3]
6	$\Diamond[(\exists x)\neg cat(x) \wedge (\forall y)cat(y)]$	1, 3-5 In$\Diamond$ E [1,2]

We cannot import [4] from line 1 because only sentences content restricted to [cat()] can be imported.

[This is redundant so you should instead prove it as lemma by formula induction (the key part is the claim about content restriction). If θ is content restricted to $\mathscr{L}, R_1, \dots R_n$ and $S_1 \dots S_m$ are relations not among $\mathcal{L}, R_1, \dots R_n$ then $\Diamond_{\mathcal{L},S_1 \dots S_m}\theta$ is content restricted to $\mathcal{L}$. Thus from $\Gamma_1 \vdash \Diamond_{\mathcal{L}}\theta$ we can deduce (using inner diamond and diamond intro) $\Gamma_1 \vdash \Diamond_{\mathcal{L}}\Diamond_{\mathcal{L},S_1 \dots S_m}\theta$ and then by diamond elim we can deduce $\Gamma_1 \vdash \Diamond_{\mathcal{L},S_1 \dots S_m}\theta$. Same trick gives you converse direction]

(◊ Ign) ◊ Ignoring. Suppose θ is content-restricted to $\mathcal{L}, R_1, \dots R_n$ and $S_1 \dots S_m$ are relations not among $\mathcal{L}, R_1, \dots R_n$. If $\Gamma_1 \vdash \Diamond_{\mathcal{L}}\theta$ then $\Gamma_1 \vdash \Diamond_{\mathcal{L},S_1 \dots S_m}\theta$. Conversely, if $\Gamma_1 \vdash \Diamond_{\mathcal{L},S_1 \dots S_m}\theta$ then $\Gamma_1 \vdash \Diamond_{\mathcal{L}}\theta$.

Remember that when a formula is content-restricted to $\mathcal{L}$, its truth depends only on facts about $\mathcal{L}$. This principle reflects this intuition by allowing one to ignore other facts.

Examples:

- It is possible, given what cats there are, that there every cat admires a distinct dog ↔ It is possible, given what cats and dolphins there are, that every cat admires a different dog.

- But NOT: It is possible, given what cats there are, that there are exactly 3 objects ↔ It is possible, given what cats and dolphins there are, that there are exactly 3 objects.

  This inference is not permitted because the claim that there are exactly 3 objects is not content restricted to any list of relations including $cats()$ but not $dolphin()$.

- And NOT: It is possible, given what cats there are, that every cat admires a distinct dog $\leftrightarrow$ It is possible, given what cats and dogs there are, that every cat admires a distinct dog.

  Here θ is content restricted to {cat, dog, admires}, but for this inference to be permitted θ would have to be content restricted to a list that didn't include the relation dog.

For each of the $\Diamond$ principles above, an analogous inference involving $\Box$ can be justified, exploiting the fact that $\Box_{\mathcal{L}}\phi$ abbreviates $\neg\Box_{\mathcal{L}}\neg\phi$. See Appendix 7.4 for details. Also like $\exists$ the choice to define $\Box_{\ldots}$ in terms of $\Diamond_{\ldots}$ rather than vice-versa was arbitrary and either choice yields the same results.

7.3 Example: Pasting Lemma

Let us now get a little experience with how these basic inference rules work together, by using them to prove the following helpful lemma.

Lemma 7.3.1. ***(P) Pasting*** *Let $\mathcal{I}$, $\mathcal{J}$ and $\mathcal{L}$ be pairwise disjoint sets of relations. If $\Diamond_{\mathcal{L}}\phi$, where ϕ is content restricted to $\mathcal{L}, \mathcal{I}$ and $\Diamond_{\mathcal{L}}\psi$, where ψ is content-restricted to $\mathcal{L}, \mathcal{J}$, then $\Diamond_{\mathcal{L}}(\phi \wedge \psi)$.*

One cannot generally infer from $\Diamond_L\phi$ and $\Diamond_L\psi$ to $\Diamond_L(\phi \wedge \psi)$; consider the case where ϕ says there are exactly 8 million things and ψ says there are exactly 9 million things. However, this principle says that one *can* make this inference when ϕ and ψ describe suitably disjoint aspects of the universe (outside of the objects satisfying relations in $\mathcal{L}$).

We can prove this lemma using the basic inference rules and axiom schemas above as follows:

Proof. Let ϕ be content restricted to $\mathcal{L}, \mathcal{I}$ and ψ to $\mathcal{L}, \mathcal{J}$, as per the antecedent.

$$\begin{array}{rlll}
1 & \Diamond_{\mathcal{L}}\phi & & [1] \\
2 & \Diamond_{\mathcal{L}}\psi & & [2] \\
3 & \Diamond\,\vert\ \underline{\phi \quad [\mathcal{L}]} & & 1,\ \text{In}\Diamond\text{I}\ [1] \\
4 & \quad\vert\ \Diamond_{\mathcal{L}}\psi & & 2,\ \text{import}\ [2] \\
5 & \quad\vert\ \Diamond_{\mathcal{L},\mathcal{I}}\psi & & 4,\ \text{Ign}\ [2] \\
6 & \quad\vert\ \Diamond\,\vert\ \underline{\psi \quad [\mathcal{L}, \mathcal{I}]} & & 5,\ \text{In}\Diamond\text{I}\ [2] \\
7 & \quad\vert\ \quad\vert\ \phi & & 3,\ \text{import}\ [1] \\
8 & \quad\vert\ \quad\vert\ \phi \wedge \psi & & 5{,}6\ \&\text{I}\ [1{,}2] \\
9 & \quad\vert\ \Diamond_{\mathcal{L}}(\phi \wedge \psi) & & 5{,}6\text{-}8\ \text{In}\Diamond\text{E}\ [1{,}2] \\
10 & \Diamond_{\mathcal{L}}(\Diamond_{\mathcal{L}}(\phi \wedge \psi)) & & 1{,}3\text{-}9\ \text{In}\Diamond\text{E}\ [1{,}2] \\
11 & \Diamond_{\mathcal{L}}(\phi \wedge \psi) & & 10,\ \Diamond\text{E}\ [1{,}2]
\end{array}$$

□

Informally, this deduction corresponds to the following reasoning:

Assume that $\Diamond_{\mathcal{L}}\phi$ and $\Diamond_{\mathcal{L}}\phi$. We can prove our claim by making two nested In$\Diamond$ arguments.

First enter the $\Diamond_{\mathcal{L}}$ context associated with $\Diamond_{\mathcal{L}}\phi$. In this context we clearly have ϕ. But we also know that $\Diamond_{\mathcal{L}}\psi$ must remain true, because it was true in our previous context and it is content restricted to $\mathcal{L}$. We can deduce from this that $\Diamond_{\mathcal{L},\mathcal{I}}\psi$ by Ignoring.

Now enter this second, interior, $\Diamond_{\mathcal{L},\mathcal{I}}$ context. Here we clearly have ψ. But we can import the fact that ϕ from the previous context, because it is content restricted to $\mathcal{L},\mathcal{I}$. So we can deduce $\phi \wedge \psi$.

Now, leaving this inner $\Diamond_{\mathcal{L},\mathcal{I}}$ context, we can conclude that $\Diamond_{\mathcal{L}}(\phi \wedge \psi)$ (because $\mathcal{L}$ is clearly a sublist of $\mathcal{L},\mathcal{I}$).

So, leaving the larger $\Diamond_{\mathcal{L}}$ context we can conclude that $\Diamond_{\mathcal{L}}(\Diamond_{\mathcal{L}}(\phi \wedge \psi))$ holds in the situation we were originally considering.

Finally, because $\Diamond_{\mathcal{L}}(\phi \wedge \psi)$ is content restricted to $\mathcal{L}$, we can use $\Diamond$E to draw the desired conclusion $\Diamond_{\mathcal{L}}(\phi \wedge \psi)$.

7.4 $\Box$ Inf. Rules

Although the $\Box$ is not an official item in our symbolism, but merely an abbreviation for $\neg\Diamond\neg$, it is often helpful to reason in terms of it. Thus we should note that the above inference rules can be used to vindicate analogous inference rules involving the $\Box$:

($\Box$ I) Box Introduction. If $\Gamma \vdash \theta$, where $\Gamma = \gamma_1 ... \gamma_m$ and for all i γ_i is content-restricted to $\mathcal{L}$ then $\Gamma \vdash \Box_{\mathcal{L}}\theta$.

As with In$\Diamond$, I will use some visually suggestive notation to keep track of inferences of this form, as follows:

1	γ	Assump [1]
2	$\Box$ $[\mathcal{L}]$	
3	γ	1, import [1]
4	...	
5	ϕ	[1]
6	$\Box_{\mathcal{L}}\phi$	2-5 $\Box$I [1]

Intuitively speaking, the forked line going from 3-6 above demarcates reasoning about what an arbitrary logically possible scenario in which all the facts about $\mathcal{L}$ (in our current context) are held fixed would have to be like.

A line ρ can be written down inside this "$\Box_{\mathcal{L}}$ introduction context" if

- $\rho = \gamma$ for some γ which is content-restricted to $\mathcal{L}$ and occurs on an earlier line in the proof in the same context as the intended conclusion of this $\Box_{\mathcal{L}}$I argument.

- ρ follows from previous lines in this $\Box_{\mathcal{L}}$ introduction context by one of the inference rules for reasoning about logical possibility presented in this paper.

One can leave $\Box_{\mathcal{L}}$ context above by going from knowledge that ϕ holds within this context to the conclusion that $\Box_{\mathcal{L}'}\phi$ holds outside it, provided that $\mathcal{L}'$ is a sublist of $\mathcal{L}$.

Proof. Suppose we have $\gamma_1 ... \gamma_m \vdash \theta$ as above. Then we can derive $\Box_{\mathcal{L}}\theta$ from Γ as follows.

1		$\gamma_1 ... \gamma_m$	[1]
2		$\Diamond_{\mathcal{L}} \neg\theta$	[2]
3	$\Diamond$	$\quad \neg\theta$	In$\Diamond$ I [2]
4		$\quad \gamma_1 ... \gamma_m$	import [Γ]
5		$\quad ...$	
6		$\quad \theta$	[2, Γ]
7		$\quad \bot$	3, 6 $\bot$ I
8		$\Diamond_{\mathcal{L}} \bot$	2,3-7 In$\Diamond$E [2,Γ]
9		$\bot$	8 $\Diamond$E [2,Γ]
10		$\neg\Diamond_{\mathcal{L}} \neg\theta$	2-9 $\neg$I [Γ]
11		$\Box_{\mathcal{L}}\theta$	[Γ]

$\Box$

($\Box$ E) Box Elimination. If $\Gamma \vdash \Box_{\mathcal{L}}\theta$ then $\Gamma \vdash \theta$

1	$\Box_{\mathcal{L}}\theta$	[Γ]
2	$\neg\Diamond_{\mathcal{L}}\neg\theta$	[Γ]
3	$\neg\theta$	Assump. [3]
4	$\Diamond_{\mathcal{L}}\neg\theta$	4 $\Diamond$ I [3]
5	$\bot$	2, 4 $\bot$ I [3,Γ]
6	$\neg\neg\theta$	3-5 ¬I [Γ]
7	θ	6 ¬E [Γ]

Chapter 8

The Formal System II: Other Inference Rules

(Cut) Cutback If $\mathcal{L} = R_1, \ldots, R_m$ is a list of relations and $\vec{x}^i = x^i_1 ... x^i_{n_i}$ where n_i is the arity of R_i, then $\Gamma \vdash (\exists x)[P(x) \wedge (\forall \vec{x}^1)(R_1(\vec{x}^1) \rightarrow P(x^1_1) \wedge ... P(x^1_{n_1})) \wedge ... (\forall \vec{x}^m)(R_m(\vec{x}^m) \rightarrow P(x^m_1) \wedge ... P(x^m_{n_m}))] \rightarrow \Diamond_{\mathcal{L},P}(\forall x)P(x)$

This axiom schema expresses the idea that if a predicate P applies to all the objects which relations in $\mathcal{L}$ apply to (and P applies to at least one thing), then it is logically possible to have a 'cut back' universe which preserves how P and relations in $\mathcal{L}$ apply, and contains no objects outside the extension of P.

[note: edited check]

(ReL) Relabeling. If $R_1 \ldots R_n$ are relations that occur in θ but not in $\mathcal{L}$, and $R'_1 \ldots R'_n$ are relations with the same arities (i.e., the arity of

R_i and R'_i are the same) that don't occur in $\mathcal{L}$ or θ, then $\Gamma \vdash \Diamond_{\mathcal{L}}\theta \leftrightarrow \Diamond_{\mathcal{L}}\theta[R_1/R'_1 \ldots R_n/R'_n]$.

This axiom schema expresses the idea that when evaluating claims about logical possibility, all relations of the same arity have the same behavior (we abstract away from their underlying behavior [alt: for the purposes of logical possibility no other facts about a relation matter]). Thus, replacing some $R \notin \mathcal{L}$ with an unused relation $R' \notin \mathcal{L}$ of the same arity cannot change the truthvalue of $\Diamond_{\mathcal{L}}\theta$.

Example: By substituting sleeps with chews we see "It is logically possible, given the facts about dogs and blankets, that every dog sleeps on a different blanket" $\Leftrightarrow$ "It is logically possible, given the facts about dogs and blankets, that every dog chews on a different blanket."

Note that sleeps and chews are both relations that are not in the list of relations being subscripted $\mathcal{L} =$ dog, blanket.

(SC) Simple Comprehension. If ψ is a sentence which contains no $\Box$s or $\Diamond$s and the relation R doesn't occur in $\mathcal{L}, \phi$ or ψ, then $\Gamma \vdash \psi \rightarrow \Diamond_{\mathcal{L}}[\psi \wedge (\forall \vec{z})(R(\vec{z}) \leftrightarrow \phi(\vec{z}))]$.

This axiom schema captures the idea that it is possible (holding fixed $\mathcal{L}$) for an otherwise unused relation R to apply to exactly those tuples $\vec{z}$ which satisfy some first order formula $\phi(\vec{z})$. Moreover, intuitively it is possible for R to be so defined without changing the truth of any sentences not containing R.

> Example: "If there is something which everyone loves, it is logically possible (given the facts about love) that there is something which everyone loves *and* happy() applies to exactly those individuals which love themselves."

Our next axiom schema, Modal Comprehension, expresses a somewhat similar idea to the Simple Comprehension Schema above. It says that one can sometimes specify a logically possible way for a relation $\boldsymbol{R}$ to apply by appealing to properties that can only be expressed [discuss: this seems a little strong] using modal operators, e.g., the property of x envying infinitely many objects.

Informally, we can specify how an (otherwise unused) relation $\boldsymbol{R}$ applies by saying that it applies to exactly those n-tuples of objects in $\boldsymbol{Ext}(\mathscr{L})$, which satisfy a certain property expressed in terms of logical possibility operators and the relations in $\mathscr{L}$. It lets us express (and recognize the truth of) claims which seem to require quantifying in, like:

> SIBLINGS: Holding fixed the facts about the relations Married(x, y) and Sibling(x, y) it is logically possible to have a relation $\boldsymbol{R}(x)$ that applies to exactly those married individuals x with more siblings than their spouse.

Note that having more siblings than one's spouse has to be cashed out in terms of the logical possibility of a surjective but not injective map from their siblings to those of their spouse. On first glance, it would appear this would require passing x (the individual for whom we wish to compare their siblings to those of their spouse) into the logical possibility operator evaluating the possibility of such a pairing. However, our language of logical possibility does not allow this kind of quantifying in.

Instead, we do this by using a special, otherwise-unused, n-place relation Q to label and preserve a choice for an n-tuple of objects in $Ext(\mathscr{L})$. We say that it is possible (fixing the $\mathscr{L}$ facts) for R to apply in such a way that, necessarily (fixing the $\mathscr{L}, R$ facts), R only relates objects in $Ext(\mathscr{L})$ and however Q chooses a unique n-tuple of objects in $Ext(\mathscr{L})$ for consideration, R applies to this n-tuple iff a certain modal claim ϕ describing the behavior of $\mathscr{L}$ and Q is true. In this case, the relevant $\mathscr{L}$ is $\text{Married}, \text{Sibling}$, and the modal sentence ϕ is $\Diamond_{\text{Married,Sibling},Q}$ $((\exists x)Q(x) \wedge (\exists y)\text{Married}(x, y)$ and $Z(\cdot, \cdot)$ is a surjective but not injective map from the siblings of x to those of y.

We can thus express the informal claims like siblings with a sentence of the following form:

$$\begin{aligned} &\Diamond_{\mathscr{L}} \Box_{\mathscr{L},R} (\exists!\, x Q(x) \rightarrow \\ &\exists x (Q(x) \wedge [R(x) \leftrightarrow x \in Ext(\mathscr{L}) \wedge \phi)] \end{aligned} \tag{8.1}$$

Since it is possible for Q to apply to any single object the necessity operator above ensures that R applies to exactly those x which have more siblings than their spouse. With this motivation in place, I can now state the Modal Comprehension Schema as follows

(MC) Modal Comprehension If

- R does not occur in $\mathcal{L}, \psi$ or ϕ
- Q does not occur in $\mathcal{L}$ or ψ

- ϕ is content restricted to $\mathcal{L}, Q$

then $\Gamma \vdash \psi \rightarrow \Diamond_{\mathcal{L}}(\psi \wedge \Box_{\mathcal{L},R}(\exists! \vec{x} Q(\vec{x}) \rightarrow (\exists \vec{x})(Q(\vec{x}) \wedge [R(\vec{x}) \leftrightarrow \vec{x} \in Ext(\mathscr{L}) \wedge \phi(\vec{x})]))]$

where $\exists! \vec{x} Q(\vec{x})$ means that Q applies to a unique n-tuple of objects.

Axiom 8.0.1 (Infinity)**.** *It is possible for a two place relation S to apply in the following successor-like way:*

- *The successor of an object is unique* $(\forall x)(\forall y)(\forall y')[S(x,y) \wedge S(x,y') \rightarrow y = y']$
- *successor is one-to-one* $(\forall x)(\forall y)(\forall x')(S(x,y) \wedge S(x',y) \rightarrow x = x')$
- *there is a unique object that has a successor and isn't the successor of anything* $(\exists! x)(\exists y)(S(x,y) \wedge (\forall y)[\neg S(y,x)])$
- *everything that is a successor has a successor* $(\forall x)[(\exists y)S(y,x) \rightarrow (\exists z)S(x,z)]$
- *S is anti-reflexive:* $(\forall x)(\forall y)[S(x,y) \rightarrow \neg S(y,x)]$

[verify somewhere this is implicitly content restricted to S...OR ADD D back in]

[You need to also be able to subscript other predicates or this is mostly useless...add an L] **(PP) Possible Powerset.** If F, C are distinct predicates and $\in$ a two-place relation, then $\Gamma \vdash \Diamond_F \mathscr{C}(C, \in, F)$.

Here $\mathscr{C}(C, \in, F)$ means that C and F are disjoint, $\in$ relates (only) objects satisfying C to objects satisfying F and:

- $\Box_{C,\in,F}(\exists x)[C(x) \wedge (\forall y)((F(y) \wedge K(y)) \leftrightarrow y \in x)]$, i.e., it's necessary that however some new predicate K applies to some objects satisfying F, there exists a corresponding 'class' C whose 'elements' are exactly the objects which F applies to.

- $(\forall y)(\forall y')(C(y) \wedge C(y') \wedge \neg y = y' \rightarrow (\exists x)\neg(x \in y \leftrightarrow x \in y')$, i.e., no two members of C contain (in the sense of $\in$) the same elements.

Intuitively, this axiom schema says that it is always possible to "add a layer of classes" to the objects satisfying some predicate F.

[must be for n+1-ary relations...in particular n needs to be able to be 0 for getting us the wrapping trick...that means explaining that when n is 0 we don't have I in there at all...or give 0 case as consequence you prove] **(Choice) Combinatorial Choice.** $\Gamma \vdash (\forall x)[I(x) \rightarrow (\exists y)R(x,y)] \rightarrow \Diamond_{I,R}[(\forall x)(\forall y)(\hat{R}(x,y) \rightarrow R(x,y)) \wedge [(\forall x)(I(x) \rightarrow (\exists! \, y)\hat{R}(x,y)])$

This axiom schema captures the same intuition as the axiom of choice in set theory. It says that if every x satisfying I is related to some y by R, then (fixing I, R) another relation $\hat{R}$ can behave like a choice function selecting a unique such y for each x.

[this last part was edited, review, DISCUSS HOW TO CLEAN THIS]

My last, and most complex, axiom schema is combinatorial replacement. One can think of it as a kind of generalization of the Pasting Lemma **??**. Crudely speaking, it takes us from the logical possibility (given some starting structure $\mathcal{L}$), of satisfying a certain formula $\phi(x)$ *for any single x* in a base collection of objects (those satisfying some I in $\mathcal{L}$), to the logical possibility of an expanded universe where *for every* object x satisfying I, there is a corresponding structure (indexed to this object x) 'within which' a version of $\phi(x)$ is true. (But note that, as with the pasting lemma, this inference is not generally safe, and only works for certain choices of ϕ to be specified below.)

For example, if we take I to be the predicate person$(\cdot)$ and the $\mathcal{L}$ to be the list person$(\cdot)$, childOf(x, y) Combinatorial Comprehension licences claims like:

> If, for any choice of a person, there could be (holding fixed the facts about people and parentage) as many ghosts as that person has children, then (holding fixed the facts about people and parentage) it could be that, for every person x, there are as many ghosts-relevant-to-x (disjoint from everyone else's ghosts) as x has children.

As before, articulating this principle can seem to require quantifying in to the $\Diamond$ of logical possibility. However, we can use the same trick (involving an otherwise unused predicate Q) to get around it, as we did when formulating modal comprehension above. Rather than talking about possible choices of an object x satisfying I, we will talk about ways (holding fixed the $\mathcal{L}$ facts) that a predicate Q could pick out a unique object satisfying I. And then rather than talking about the logical possibility of (preserving the $\mathcal{L}$ while) making a formula $\phi(x)$ true, we will talk about the possibility of (preserving the $\mathcal{L}, Q$ facts while) making a sentence ϕ which talks about the unique object satisfying Q true.

(CR) Combinatorial Replacement (aka the Chia Pet Axiom Schema) If

- $\mathcal{L}$ is a list of relations which contains the predicate $\boldsymbol{I}$ but not $\boldsymbol{Q}$ or $\boldsymbol{R}_1 ... \boldsymbol{R}_n$

- ϕ is content-restricted $\mathcal{L}, \boldsymbol{Q}, \boldsymbol{R}_1 \ldots \boldsymbol{R}_n$. (where $\boldsymbol{P}$, $\boldsymbol{R}_1 \ldots \boldsymbol{R}_n$ and $\mathcal{L}$ share no relations)

- $\hat{\boldsymbol{R}}_1 \ldots \hat{\boldsymbol{R}}_n$ are otherwise unused relations such that if $\boldsymbol{R}_i$ is an n-place relation $\hat{\boldsymbol{R}}_i$ is an $n+1$ place relation.

Let $\Psi(x)$ be the formula

$$\bigwedge_{1 \leq i \leq n} (\forall \vec{v})(\boldsymbol{R}_i(\vec{v}) \leftrightarrow \hat{\boldsymbol{R}}_i(\vec{v}, x))$$

asserting that $\hat{\boldsymbol{R}}_i$ with x inserted into the last place behaves exactly the same as $\boldsymbol{R}_i$

Let $\rho(x, y)$ be the following formula

$$\bigvee_{\substack{1 \leq i \leq n \\ 1 \leq j \leq l_i}} (\exists z_1) \ldots, (\exists z_{j-1}), (\exists z_{j+1}), \ldots, (\exists z_{l_i}) \hat{\boldsymbol{R}}_i(z_1, \ldots, z_{j-1}, x, z_{j+1}, \ldots, z_{l_i}, y)$$

In other words $\rho(x, y)$ asserts that x appears in some tuple ending with y satisfying some $\hat{\boldsymbol{R}}_i$

then

$$\Gamma \vdash \Box_{\mathcal{L}}[\exists!\, x(Q(x) \wedge I(x)) \to \Diamond_{\mathcal{L},Q}\phi] \to$$
$$\Diamond_{\mathcal{L}} \left[(\forall x)(\forall y)(\forall y') \left[(\neg y = y' \wedge \rho(x,y) \wedge \rho(x,y') \to x \in Ext(\mathcal{L})\right] \wedge\right.$$
$$\left.\Box_{\mathcal{L},\hat{R}_1 \ldots \hat{R}_n}[\exists!\, x(Q(x) \wedge I(x)) \wedge \exists x(Q(x) \wedge I(x) \wedge \Psi(x)) \to \phi\right]$$

To see why this is intuitive, think of $\mathcal{L}$ as specifying an original structure we are starting with. The antecedent "$\Box_{\mathcal{L}}[\exists!\, x(P(x) \wedge I(x)) \to \Diamond_{\mathcal{L},P}\phi]$" says that this structure ensures that however the predicate Q to selects a single object within I from this structure, it would be logically possible (preserving these $\mathcal{L}, Q$ facts) for the relations $R_1 ... R_n$ to apply so as to make a sentence ϕ (which only talks about how the unique object satisfying Q relates to the $\mathcal{L}, R_1 ... R_n$ structure) true.

The conclusion says that this original $\mathcal{L}$ structure could be extended by simultaniously adding objects satisfying relations $\hat{R}_1 ... \hat{R}_n$, which pair each x within I with choice for how the relations $R_1 ... R_n$, could apply to this expanded domain which has two features. First, it's logically necessary that, however Q chooses a single x in I, if $R_1 ... R_n$ apply to exactly those objects satisfying $\hat{R}_1(..., x) ... \hat{R}_n(...., x)$ then the sentence $\phi[\mathcal{L}, Q, R_1 ... Rn]$ is true. And second, the expanded structures of objects satisfying $R_1 ... R_n$ for different x satisfying I are disjoint, only overlapping (at most) the original structure $\mathcal{L}$.

Chapter 9

Example: Lemmas about Well-Orderings

To give a more visceral sense of how the above axioms work, I will now present two proofs in my system of logical possibility that that mirror results in set theory (which can be found in elementary texts like [4]).

In later chapters I will present proofs in a more informal style. However I will try to make it clear to the reader how each of these proofs can be expanded out into the fully formal proofs demonstrated below.

In what follows, I will sometimes use $\exists_F x \phi(x)$ to abbreviate $\exists x F(x) \wedge \phi$ and $\forall_F x \phi(x)$ to abbreviate $(\forall x)(F(x) \rightarrow \phi(x))$.

9.1 Lemma A

Jech's version of the first lemma I am going to prove says the following:

"If $(W, <)$ is a well-ordered set and $f : W \rightarrow W$ is an increasing function, then $x < f(x)$ for each $x \in W$."

We can express something like the same idea in the language of logical possibility by making the following definitions:

Definition 9.1.1. A two-place relation $<$ **well-orders** the objects which satisfy F iff

- $<$ linearly orders the objects which satisfy F
  - $<$ is total

    $$(\forall x)(\forall y)(x < y \vee y < x)$$

  - $<$ is transitive

    $$(\forall x)(\forall y)(\forall z)(x < y \wedge y < z \rightarrow x < z)$$

  - $<$ is total on F

    $$(\forall x)(\forall y)(F(x) \wedge F(y) \rightarrow x < y \vee y < x)$$

- F, $<$ satisfy the Least Element Condition: If some element satisfying F also satisfies G then there is a least element in F satisfying G:

  $$\Box_{F,<}[(\exists x)(F(x) \wedge G(x)) \rightarrow (\exists y)(F(y) \wedge G(y) \wedge (\forall z)(F(z) \wedge G(z) \rightarrow y < z))]$$

Definition 9.1.2. A two place relation R **behaves like an increasing function** from some $F, <$ to $G, <'$ if

- $(\forall x)[F(x) \rightarrow (\exists y)(G(y) \wedge R(x, y))]$
- $(\forall x)(\forall y)[R(x, y) \rightarrow (\forall z)(R(x, z) \rightarrow y = z)]$

- $(\forall x)(\forall y)(\forall x')(\forall y')[x < y \wedge R(x, x') \wedge R(y, y') \rightarrow x' < y']$.

I will sometimes abbreviate the above claim as $F <\mapsto_{R,inc} G <'$ since it says that R behaves like an increasing function which maps from $F <$ (the Fs under $<$) to $G <'$ (the Gs under $<'$).

As usual, I will sometimes abbreviate the claim that $x < y \wedge \neg x = y$ as $x < y$.

Thus we can write a version of Jech's Lemma follows:

> **Lemma A:** "If R behaves like an increasing function from $F, <$ to $F, <$ and the objects satisfying F are well ordered by $<$, then $(\forall x)(\forall y)[F(x) \wedge F(y) \wedge R(x, y) \rightarrow \neg y < x]$ "

Proof. To prove Lemma A, we will use essentially the same reasoning which Jech uses to prove his set theoretic version of this claim. He says: Let F be well-ordered by $<$ and let R behave like an increasing function from F to F $<$. Suppose the conclusion of the lemma does not hold. Then by the fact that F is well-ordered by $<$, there must be a least x such that $f(x) < x$, (i.e. $(\exists y)(R(x, y) \wedge y < x)$. Consider $y = f(x)$. Because $y < x$ and R is increasing, $f(y) < f(x) = y$. This contradicts the claim that x is the $<$-least z such that $F(z) \wedge f(z) < z$.

To mimic this reasoning, we first suppose that $<$ well-orders the Fs and R behaves like an increasing function from the Fs to the Fs. Now we want to consider the collection of objects satisfying F which R maps below themselves (just as in Jech's proof). Our main difficulty will be in using the fact that the Fs are well-ordered by $<$ (in the sense specified by the modal definition of well ordering above) to show that if there are any such objects, there must be an $<$-least one.

By Simple Comprehension it is possible (while holding fixed the facts about how the relations $\boldsymbol{F}, \boldsymbol{R}$ and $<$ indicated in curly braces apply) for the otherwise-unused predicate $\boldsymbol{G}$ to apply to exactly those objects satisfying $\boldsymbol{F}$ and which $\boldsymbol{R}$ maps below themselves. Within this context, our initial assumptions that $<$ well-orders the $\boldsymbol{F}s$ and $\boldsymbol{R}$ behaves like an increasing function from the $\boldsymbol{F}s$ to the $\boldsymbol{F}s$ must remain true (because they are content restricted to $\boldsymbol{F}, \boldsymbol{R}$ and $<$). So we can use the fact that the $\boldsymbol{F}, <$ satisfy the Least Element condition, i.e. $\Box_{\boldsymbol{F},<}$(if $\boldsymbol{G}$ applies to any object satisfying $\boldsymbol{F}$, it applies to an $<$-least such object) to deduce the needed claim. So we have: if $\boldsymbol{R}$ maps any object satisfying $\boldsymbol{F}$ below itself, then then $\boldsymbol{G}$ applies to exactly these objects, and there is a $<$-least object satisfying $\boldsymbol{G}$.

Once we have this fact, we can derive the truth of the desired claim that $(\forall x)(\forall y)[\boldsymbol{F}(x) \wedge \boldsymbol{F}(y) \wedge \boldsymbol{R}(x, y) \rightarrow \neg y < x]$ holds true *within this special context* by exactly the same first order reasoning which Jech uses. Suppose for contradiction that there were some such x that got mapped below itself (in the sense above). Then we'd have a y such that $\boldsymbol{R}(x, y) \wedge y < x$. By the fact that $\boldsymbol{R}$ is a one-to-one increasing function, $y < x$ and $\boldsymbol{R}$ maps x to y, we know that $\boldsymbol{R}$ must map y to something strictly less than y[1]. But then $\boldsymbol{G}$ must also apply to y, which contradicts the claim that x is the $<$-least object satisfying $\boldsymbol{G}$.

Thus we know that $(\forall x)(\forall y)[\boldsymbol{F}(x) \wedge \boldsymbol{F}(y) \wedge \boldsymbol{R}(x, y) \rightarrow \neg y < x]$ holds true

[1]It must map y to something $< y$ because it is increasing and something $\neq y$ because it is one-to-one and $\boldsymbol{R}(x, y)$ and $\neg x = y$.

within the special modal scenario being considered above. However, we can note the above claim is content-restricted to $\boldsymbol{F}, \boldsymbol{R}, <$. Thus we can infer from the mere fact that it could be true (while holding fixed the behavior of $\boldsymbol{F}, \boldsymbol{R}$ and $<$) to the conclusion that it is actually true.

Now the desired conclusion, and the overall conditional to be proved follows immediately.

1	$<$ well-orders the $\boldsymbol{F}s \wedge \boldsymbol{F} < \mapsto_{R,inc} \boldsymbol{F} <$	[1]
2	$\Diamond_{F,<,R} \forall z[\boldsymbol{G}(z) \leftrightarrow \exists_F y \boldsymbol{R}(z,y) \wedge y < z]$	SC
3	$\Diamond$ $\forall z[\boldsymbol{G}(z) \leftrightarrow \exists_F y \boldsymbol{R}(z,y) \wedge y < z]$ $\{\boldsymbol{F}, <, \boldsymbol{R}\}$	In$\Diamond$ I
4	$<$ well-orders the $\boldsymbol{F}s \wedge \boldsymbol{F} < \mapsto_{R,inc} \boldsymbol{F} <)$	2 import [1]
5	$\Box_{F,<}(\exists_F x \boldsymbol{G}(x) \rightarrow \exists_F x[\boldsymbol{G}(x) \wedge (\forall_F z \boldsymbol{G}(z) \rightarrow x < z)])$	4 FOL [1]
6	$\exists x(\boldsymbol{F}(x) \wedge \boldsymbol{G}(x)) \rightarrow (\exists_F x[\boldsymbol{G}(x) \wedge (\forall_F z \boldsymbol{G}(z) \rightarrow x < z)]$	5 $\Box$ E [1]
7	$\neg[\exists_F x \exists_F y(\boldsymbol{R}(x,y) \wedge y < x)]$	3,4,6 FOL [1,4]
8	$\Diamond_{F,<,R} \neg[\exists_F x \exists_F y \boldsymbol{R}(x,y) \wedge y < x]$	2-7 In$\Diamond$E [1]
9	$\neg[\exists_F x \exists_F y \boldsymbol{R}(x,y) \wedge y < x]$	8 $\Diamond$E [1]
10	$\forall_F x \forall_F y(\boldsymbol{R}(x,y) \rightarrow \neg y < x)$	9 FOL [1]
11	$\forall_F x \exists_F y \boldsymbol{R}(x,y) \wedge (y = x \vee y > x)$	1, 10 FOL [1]

□

9.2 Lemma B

"No well-ordered set is isomorphic to an initial segment of itself"

Definition 9.2.1. I will say that **the $R_1, \dots R_m$ are isomorphic to the $\langle R'_1, \dots R'_m \rangle$ under some relation Z** (henceforth written $\langle R_1, \dots R_m \rangle \cong_Z \langle R'_1, \dots R'_m \rangle$) iff:

- Z behaves like a bijection between the domain of objects in $Ext(R_1, \dots R_m)$ and the domain of objects in $Ext(R'_1, \dots R'_n)$. More formally:
  - $(\forall x)(x \in Ext(R_1, \dots R_m) \rightarrow (\exists!\, ys.t.Z(x, y)) \wedge y \in Ext(R'_1, \dots R'_m)])$ i.e., Z is functional over $Ext(R_1, \dots R_m)$ with a range within $Ext(R'_1, \dots R'_m)$ and
  - $(\forall y)(y \in Ext(R'_1, \dots R'_m) \rightarrow (\exists!\, xs.t.Z(x, y) \wedge y \in Ext(R_1, \dots R_m)]),$ i.e., Z maps one-to-one and onto all of $Ext(R'_1, \dots R'_m)$
- Z applies in a way that respects each R_i, i.e., $(\forall \vec{x})(\forall \vec{y})[Z(x_1, y_1) \wedge \dots \wedge Z(x_n, y_n) \rightarrow (R_i(\vec{x}_i)) \leftrightarrow R'_i(\vec{y}_i))]$, where if R_i is an n-place relation then $\vec{x}_i = x_1, \dots x_n$ and $\vec{y}_i = y_1, \dots y_n$

We can then state a modal version of this claim as follows.

Claim to Prove: If the objects satisfying F are well-ordered by $<$ then $\neg\Diamond_{F,<}(\exists x)[F(x) \wedge \langle F; > \rangle \cong_R \langle G; > \rangle \wedge \forall z(G(z) \leftrightarrow [F(z) \wedge z < x])]$

Proof. Assume that the objects satisfying F are well-ordered by $<$. Suppose for contradiction that $\Diamond_{F,<}[(\exists x)(F(x) \wedge (\forall z)(G(z) \leftrightarrow F(z) \wedge z < x) \wedge \langle F, > \rangle \cong_R \langle G, > \rangle)]$. Consider any such scenario. The fact that the objects satisfying F are well-ordered by $<$ must remain true in this scenario (because it is content-restricted to F, G, R and $>$). By first order logic and unpacking definitions we can deduce that R therefore behaves like an increasing function from $F, <$ to $F, <$ (the key fact is that R must respect $<$).

Now, to get contradiction, we can copy over Lemma A (we have just seen that it can be proved from empty premises, so can we re-prove it as needed, within any $\Box$ or $\Diamond$ context) and derive that R does not map any object satisfying F strictly below itself. On the other hand, we know there is an object x satisfying F which is $>$ all objects satisfying G and that $\langle F, > \rangle \cong_R \langle G, > \rangle$. It follows from this by simple first order logic that R maps the any such x to a some object $y < x$. Thus contradiction/the false ($\bot$) would have to obtain in the (supposedly) logically possible scenario under consideration.

Finally, we can export this $\bot$ to our original situation (remembering that the contradiction symbol $\bot$ is content-restricted to every list of relations) and thereby complete our proof. Informally, this corresponds to reasoning that if it were $\Diamond_{F,<}[(\exists x)(F(x) \wedge (\forall z)(G(z) \leftrightarrow F(z) \wedge z < x) \wedge \langle F, > \rangle \cong_R \langle G, > \rangle)]$ then it would also be $\Diamond_{F,<}\bot$, which is false, so the original $\Diamond_{F,<}$ claim cannot be true.

1	F is well-ordered by $<$	[1]
2	$\Diamond_{F,<}[(\exists x)(F(x) \wedge (\forall z)(G(z) \leftrightarrow F(z) \wedge z < x) \wedge \langle F;>\rangle \cong_R \langle G;>\rangle)]$	[2]
3	$\Diamond$ $(\exists x)(F(x) \wedge (\forall z)(G(z) \leftrightarrow F(z) \wedge z < x) \wedge \langle F;>\rangle \cong_R \langle G;>\rangle)\{F,<\}$	In$\Diamond$I [2]
4	F is well-ordered by $<$	1 import [1]
5	R behaves like an increasing function on F	4 FOL [2]
6	Well-Ord.: F $<$ $\wedge R$ Inc. Func.:$F \rightarrow [\forall_F x \forall_F y R(x,y) \rightarrow \neg y < x]$	lemma A
7	$\forall_F x \forall_F y(R(x,y) \rightarrow \neg y < x)$	6,7,8 FOL [1,2]
8	$(\exists y)(R(x,y) \wedge G(y))$	5 FOL [5]
9	$(\exists x)(\exists y)(F(x) \wedge F(y) \wedge R(x,y) \wedge y < x)$	5,10 FOL [2,5]
10	$\bot$	9,11 FOL[1,2]
11	$\Diamond_{F,<}(\bot)$	2,3-10 In$\Diamond$E [1,2]
12	$\bot$	11 $\Diamond$E [1,2]
13	$\neg\Diamond_{F,<}[(\exists x)(F(x) \wedge (\forall z)(G(z) \leftrightarrow F(z) \wedge z < x) \wedge \langle F,>\rangle \cong_R \langle G,>\rangle)]$	$\neg$I [1,2]

So $\vdash$ F is well-ordered by $< \rightarrow \neg\Diamond_{F,<}[\exists x F(x) \wedge \Diamond_{F,<}\langle F;>\rangle \cong_R \langle G;>\rangle \wedge \forall z(G(z) \leftrightarrow [F(z) \wedge z < x])]$

□

Chapter 10

Conclusion

10.1 Review

10.2 A Motivation for Realism about GCH

10.3 Towards a Response to the Access Problem

[The access problem for (pure) facts in $\mathscr{L}$ is relatively tractable:

-Reasoning about logical possibility applies to everything, so there's some hope that we could learn general laws of logical possibility from dealing with other thigns, and then apply them to set theroy, much as we could learn law of physcial possibility from experiements on the earth with pendulums etc. and hten apply them in space.

-the practical helpfulness of reasoning about logical possibility extends to reasoning about logically possible extendability. It's useful to recognize which hypotheses can be disagreded immediately as logically impossible.

And its helpful to be able to ask questions like: is there any logically distribution of troops/meat, such that it would be logically impossible given structural facts about that distribution for the enemy to attack us in such a way that no troops can get there in more than a days march, or such that no coalition can be built such that everyone in the colaition can complain tht they are stiffed in such a nd such an intuitively moving way that no-one outside of the coalition is.]

Part II

Justifying Set Theoretic Practice

Chapter 11

Useful Corollaries to Axioms

11.1 Diamond Simplification Lemmas

Lemma 11.1.1. ***Basic Diamond Simplification*** $\vdash \Diamond_{\mathcal{L}}(\Diamond_{\mathcal{L},R_1}(\phi)) \rightarrow \Diamond_{\mathcal{L}}\phi$

Proof. Suppose $\Diamond_{\mathcal{L}}(\Diamond_{\mathcal{L},R1}(\phi))$. First we enter the outer $\Diamond_{\mathcal{L}}$ context, beginning an In$\Diamond$ argument. Since we have $\Diamond_{\mathcal{L},R_1}(\phi)$ in this context, we can apply ignoring to deduce $\Diamond_{\mathcal{L}}(\phi)$. Thus, leaving the above special context we have $\Diamond_{\mathcal{L}}(\Diamond_{\mathcal{L}}(\phi))$. Now the inside statement is content-restricted to $\mathcal{L}$, so by $\Diamond$E we can infer from its logical possibility (given the facts about $\mathcal{L}$ to its actuality). This gives us $\Diamond_{\mathcal{L}}\phi$, as desired.

1	$\Diamond_{\mathcal{L}}(\Diamond_{\mathcal{L},R_1}(\phi))$	[1]
2	$\Diamond$ $\Diamond_{\mathcal{L},R_1}(\phi)$ $[\mathcal{L}]$	1, In$\Diamond$I [1]
3	$\Diamond_{\mathcal{L}}(\phi)$	2, Ign I [1]
4	$\Diamond_{\mathcal{L}}(\Diamond_{\mathcal{L}}\phi)$	1,2-3 In$\Diamond$E [1]
5	$\Diamond_{\mathcal{L}}\phi$	4 $\Diamond$ E [1]

$\square$

Lemma 11.1.2. ***Diamond Collapsing:*** *If ϕ_2 and θ are content restricted to $\mathcal{L}_1 \cup \mathcal{L}_2$ and ϕ_1 is content restricted to $\mathcal{L}_0 \cup \mathcal{L}_1$, and $\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2$ disjoint, then we have*

$$\vdash \Diamond_{\mathcal{L}_0}(\phi_1 \wedge \Diamond_{\mathcal{L}_1}(\phi_2 \wedge \theta)) \leftrightarrow \Diamond_{\mathcal{L}_0}(\phi_1 \wedge \phi_2 \wedge \theta)$$

Proof. LTR direction:

Assume $\Diamond_{\mathcal{L}_0}(\phi_1 \wedge \Diamond_{\mathcal{L}_1}(\phi_2 \wedge \theta))$. Enter the $\Diamond_{\mathcal{L}_0}$ context. We have $\Diamond_{\mathcal{L}_1}(\phi_2 \wedge \theta)$. Because $\phi_2 \wedge \theta$ is content restricted to $\mathcal{L}_1, \mathcal{L}_2$, we can use ignoring to turn this into $\Diamond_{\mathcal{L}_0,\mathcal{L}_1}(\phi_2 \wedge \theta)$. Now enter this $\Diamond_{\mathcal{L}_0,\mathcal{L}_1}$ context. We can import ϕ_1 because it is content restricted to $\mathcal{L}_0, \mathcal{L}_1$. Thus we can deduce $\phi_1 \wedge \phi_2 \wedge \theta$.

Leaving this $\Diamond$ context (completing our inner $\Diamond$ argument), we have $\Diamond_{\mathcal{L}_0,\mathcal{L}_1}\phi_1 \wedge \phi_2 \wedge \theta$. Hence we can deduce $\Diamond_{\mathcal{L}_0}\phi_1 \wedge \phi_2 \wedge \theta$ by Ign. Noting that this latter claim is content-restricted to $\mathcal{L}_0$ lets us complete our larger $\Diamond$E argument by pulling the fact that $\Diamond_{\mathcal{L}_0}(\phi_1 \wedge \phi_2 \wedge \theta)$ outside of the outer $\Diamond_{\mathcal{L}_0}$ context.

RTL direction:

Conversely, suppose that $\Diamond_{\mathcal{L}_0}(\phi_1 \wedge \phi_2 \wedge \theta)$. Enter this $\Diamond_{\mathcal{L}_0}$ for Inn$\Diamond$. By $\Diamond$I we can infer from $\phi_2 \wedge \theta$ to $\Diamond_{\mathcal{L}_0}(\phi_2 \wedge \theta)$. Thus we have $\phi_1 \wedge \Diamond_{\mathcal{L}_0}(\phi_2 \wedge \theta)$ and completing our In$\Diamond$ gives $\Diamond_{\mathcal{L}_0}(\phi_1 \wedge \Diamond_{\mathcal{L}_1}(\phi_2 \wedge \theta))$ as desired.

□

11.2 □ Ignoring

(□ Ign) □ Ignoring. If θ is content-restricted to $\mathcal{L}, R_1, \ldots R_n$ and $S_1 \ldots S_m$ are relations not among $\mathcal{L}, R_1, \ldots R_n$ then $\vdash \Box_{\mathcal{L}, S_1 \ldots S_m}\theta \leftrightarrow \Box_{\mathcal{L}}\theta$.

1	$\Box_{\mathcal{L}}\theta$	[1]
2	$\neg\Diamond_{\mathcal{L}}\neg\theta$	[1]
3	$\Diamond_{\mathcal{L}}\neg\theta \leftrightarrow \Diamond_{\mathcal{L},S_1\ldots S_m}\neg\theta$	Ign$\Diamond$
4	$\neg\Diamond_{\mathcal{L},S_1\ldots S_m}\neg\theta$	2,3 FOL [1]
5	$\Box_{\mathcal{L},S_1\ldots S_m}\theta$	[1]
6	$\Box_{\mathcal{L}}\theta \rightarrow \Box_{\mathcal{L},S_1\ldots S_m}\theta$	5 $\rightarrow$I
7	$\Box_{\mathcal{L},S_1\ldots S_m}\theta$	[7]
8	$\neg\Diamond_{\mathcal{L},S_1\ldots S_m}\neg\theta$	[7]
9	$\neg\Diamond_{\mathcal{L}}\neg\theta$	3,8 FOL [7]
10	$\Box_{\mathcal{L},S_1\ldots S_m}\theta \rightarrow \Box_{\mathcal{L}}\theta$	9 $\rightarrow$I
11	$\Box_{\mathcal{L}}\theta \leftrightarrow \Box_{\mathcal{L},S_1\ldots S_m}\theta$	6,10 FOL

11.3 $\Box$ Collapsing Lemma

If ϕ_2 and θ are content restricted to $\mathcal{L}_1, \mathcal{L}_2$ and ϕ_1 is content restricted to $\mathcal{L}_0, \mathcal{L}_1$, then we have

$\vdash \Box_{\mathcal{L}_0}(\phi_1 \rightarrow \Box_{\mathcal{L}_1}(\phi_2 \rightarrow \theta)) \leftrightarrow \Box_{\mathcal{L}_0}(\phi_1 \wedge \phi_2 \rightarrow \theta)$

LTR direction:

Assume $\Box_{\mathcal{L}_0}(\phi_1 \rightarrow \Box_{\mathcal{L}_1}(\phi_2 \rightarrow \theta))$.

To prove that $\Box_{\mathcal{L}_0}(\phi_1 \wedge \phi_2 \rightarrow \theta)$, we consider an arbitrary scenario in which $\phi_1 \wedge \phi_2$ (and the $\mathcal{L}_0$ facts are held fixed).[1] Our initial assumption that $\Box_{\mathcal{L}_0}(\phi_1 \rightarrow \Box_{\mathcal{L}_1}(\phi_2 \rightarrow \theta))$ is content restricted to $\mathcal{L}_0$, so it must remain true in this scenario. But what is necessary must be actual, so by □E we can infer $\phi_1 \rightarrow \Box_{\mathcal{L}_1}(\phi_2 \rightarrow \theta)$. Combining this with our knowledge that ϕ_1 (in the scenario now under consideration), gives $\Box_{\mathcal{L}_1}(\phi_2 \rightarrow \theta)$. Again, what is necessary is actual, so we have $(\phi_2 \rightarrow \theta)$, and hence we can derive that θ.

Now, discharging our assumption for →I gives us $\phi_1 \wedge \phi_2 \rightarrow \theta$. And since we considered an arbitrary situation in which the facts about $\mathcal{L}_0$ were held fixed, we have $\Box_{\mathcal{L}_0}(\phi_1 \wedge \phi_2 \rightarrow \theta)$ as desired, by □I.

[1] That is to say, we enter a □I context which holds fixed $\mathcal{L}_0$ and assume for →I that $\phi_1 \wedge \phi_2$.

1		$\Box_{\mathcal{L}_0}(\phi_1 \to \Box_{\mathcal{L}_1}(\phi_2 \to \theta))$	[1]
2	$\Box$	$[\mathcal{L}_0]$	
3		$\phi_1 \wedge \phi_2$	[3]
4		$\Box_{\mathcal{L}_0}(\phi_1 \to \Box_{\mathcal{L}_1}(\phi_2 \to \theta))$	1, import [1]
5		$\phi_1 \to \Box_{\mathcal{L}_1}(\phi_2 \to \theta)$	4 $\Box$E [1]
6		$\Box_{\mathcal{L}_1}(\phi_2 \to \theta)$	3,5 FOL [1,3]
7		$\phi_2 \to \theta$	6 $\Box$E [1,3]
8		θ	3,7 FOL [1,3]
9		$\phi_1 \wedge \phi_2 \to \theta$	3,8 $\to$I [1]
10		$\Box_{\mathcal{L}}(\phi_1 \wedge \phi_2 \to \theta)$	2-5 $\Box$I [1]

RTL direction:

Conversely, assume $\Box_{\mathcal{L}_0}(\phi_1 \wedge \phi_2 \to \theta)$

To prove that $\Box_{\mathcal{L}_0}(\phi_1 \to \Box_{\mathcal{L}_1}(\phi_2 \to \theta))$, we consider an arbitrary scenario in which ϕ_1 and the $\mathcal{L}_0$ facts are held fixed. Our initial assumption above is content-restricted to $\mathcal{L}_0$, so it must remain true in this scenario.

Then we consider a further arbitrary scenario in which ϕ_2 (while the application of $\mathcal{L}_0, \mathcal{L}_1$ in the scenario above is held fixed). Since ϕ_1 held true in the previous scenario, and it is content restricted to $\mathcal{L}_0, \mathcal{L}_1$ it must remain true in this second scenario. Thus we have $\phi_1 \wedge \phi_2$. Similarly, since our initial assumption that $\Box_{\mathcal{L}_0}(\phi_1 \wedge \phi_2 \rightarrow \theta)$ was true in the previous scenario and it is content-restricted to $\mathcal{L}_0, \mathcal{L}_1$, it must also remain true in the scenario currently under consideration. And since what is necessary is actual, we can derive $\phi_1 \wedge \phi_2 \rightarrow \theta$. Putting this together with $\phi_1 \wedge \phi_2$ gives us that θ is true in the scenario under consideration.

Now in the previous paragraph, we have shown that an arbitrary scenario in which the $\mathcal{L}_0, \mathcal{L}_1$ facts from our first scenario are preserved and ϕ_2 holds true must also be one in which θ. Thus we know that our first scenario was one in in which $\Box_{\mathcal{L}_0,\mathcal{L}_1}(\phi_2 \rightarrow \theta)$, by conditional proof and then □I. And since $\phi_2 \rightarrow \theta$ is content-restricted to $\mathcal{L}_1$, we can use (the □ version of) ignoring deduce that $\Box_{\mathcal{L}_1}(\phi_2 \rightarrow \theta)$.

Thus we have shown that an arbitrary scenario in which ϕ_1 is true and the $\mathcal{L}_0$ facts are held fixed must be one in which $\Box_{\mathcal{L}_1}(\phi_2 \rightarrow \theta)$. From this it follows by □I and conditional proof that $\Box_{\mathcal{L}_0}(\phi_1 \rightarrow \Box_{\mathcal{L}_1}(\phi_2 \rightarrow \theta))$ as desired.

1	$\Box_{\mathcal{L}_0}(\phi_1 \wedge \phi_2 \to \theta)$	assump. [1]
2	$\Box$ $[\mathcal{L}_0]$	
3	$\Box_{\mathcal{L}_0}(\phi_1 \wedge \phi_2 \to \theta)$	1 import [1]
4	ϕ_1	assump. [3]
5	$\Box$ $[\mathcal{L}_0, \mathcal{L}_1]$	
6	ϕ_2	assump. [6]
7	ϕ_1	4 import [3]
8	$\phi_1 \wedge \psi$	6, 7 FOL [3,6]
9	$\Box_{\mathcal{L}_0}(\phi_1 \wedge \phi_2 \to \theta)$	3 import [1]
10	$\phi_1 \wedge \phi_2 \to \theta$	9 $\Box$E [1]
11	θ	8,10 FOL [1,3,6]
12	$\phi_2 \to \theta$	6,11 $\to$ I [1,3]
13	$\Box_{\mathcal{L}_0,\mathcal{L}_1}(\phi_2 \to \theta)$	5-12 $\Box$I [1,3]
14	$\Box_{\mathcal{L}_1}(\phi_2 \to \theta)$	13 $\Box$Ign [1,3]
15	$\phi_1 \to \Box_{\mathcal{L}_1}(\phi_2 \to \theta)$	3,14 $\to$I [1]
16	$\Box_{\mathcal{L}_0}(\phi_1 \to \Box_{\mathcal{L}_1}(\phi_2 \to \theta))$	2-15 $\Box$I [1]

Putting these two arguments together in the obvious first order logical way gives us $\Box_{\mathcal{L}_0}(\phi_1 \to \Box_{\mathcal{L}_1}(\phi_2 \to \theta)) \leftrightarrow \Box_{\mathcal{L}_0}(\phi_1 \wedge \phi_2 \to \theta)$.

11.4 Box Relabeling

Lemma 11.4.1. ***Box Relabling*** *If $R_1 \dots R_n$ are relations that occur in θ but not in $\mathcal{L}$, and $R'_1 \dots R'_n$ are relations with the same arities (i.e., the arity of R_i and R'_i are the same) that don't occur in $\mathcal{L}$ or θ, then $\Gamma \vdash \Box_{\mathcal{L}}\theta \leftrightarrow \Box_{\mathcal{L}}\theta[R_1/R'_1 \dots R_n/R'_n]$.*

Proof. We can prove this straighforwardly from Relabling and the fact that $\Box$ abbreviates $\neg\Diamond\neg$

1	$\Diamond_{\mathcal{L}}\neg\theta \leftrightarrow \Diamond_{\mathcal{L}}\neg\theta[R_1/R'_1 \dots R_n/R'_n]$	ReL
2	$\neg\Diamond_{\mathcal{L}}\neg\theta \leftrightarrow \neg\Diamond_{\mathcal{L}}\neg\theta[R_1/R'_1 \dots R_n/R'_n]$	1, Fol
3	$\Box_{\mathcal{L}}\theta \leftrightarrow \Box_{\mathcal{L}}\theta[R_1/R'_1 \dots R_n/R'_n]$	by def of box

$\Box$

11.5 Multiple Definitions Lemma

Lemma 11.5.1. ***Multiple Definition Lemma****: Often we will want to make a chain of explicit definitions – to using Simple Comprehension or Modal Comprehension, Possible Powerset or Choice to specify the application of a series of relations $R_1 ... R_n$ in turn. Thus we have*

- ϕ

- $\Diamond_{\mathcal{L}}\psi_1$, *where* ψ_1 *specifies a way that* $\boldsymbol{R}_1$ *could apply in terms of* $\mathcal{L}$ *(so* ψ_1 *content-restricted to* $\mathcal{L}, \boldsymbol{R}_1$*),*
- *inside this* $\Diamond$ *context* $\Diamond_{\mathcal{L},R_1}\psi_2$ *where* ψ_2 *specifies a way that* $\boldsymbol{R}_2$ *could apply in terms of* $\mathcal{L}, \boldsymbol{R}_1$ *(so* ψ_2 *content-restricted to* $\mathcal{L}, \boldsymbol{R}_1, \boldsymbol{R}_2$*)*
- *etc.*

And we can hence conclude that $\Diamond_{\mathcal{L}}(\phi \wedge \psi_1 \wedge \Diamond_{\mathcal{L},R_1}(\psi_2 \wedge \Diamond_{\mathcal{L},R_1,R_2}(\psi_3 \wedge ...)))$.

In such cases we can infer the logical possibility of a single scenario $\Diamond_{\mathcal{L}}(\phi \wedge \psi_1 \wedge ...\psi_n)$

Proof. The desired conclusion follows immediately by repeated application of FOL to suitable instances of the $\Diamond$-collapsing lemma above. □

11.6 Simplified Choice

Simple Choice $\vdash (\exists x)P(x) \to \Diamond_P(\exists x(P(x) \wedge P'(x) \wedge (\forall y)[P'(y) \to x = y])$

Suppose for →I, that $(\exists x)P(x)$.

We can use the Possible Powerset axiom schema to get the possibility that $class()$and $\in$ behave like a layer of classes over the objects satisfying $\boldsymbol{P}$ and there is an object which behaves like the $\varnothing$ alongside the objects satisfying $\boldsymbol{P}$. Enter this $\Diamond_P$-context and use Simple Comprehension to set $(\forall x)(F(x) \leftrightarrow x = \varnothing)$[2] and then (entering this $\Diamond_{P,class,\in}$-context), the possibility that $\boldsymbol{R}$ relates $\varnothing$ to each object satisfying $\boldsymbol{P}$ [i.e., $(\forall x)(\forall y)R(x,y) \leftrightarrow x = \varnothing \wedge P(y)$]. Enter that $\Diamond_{P,class,\in,F}$-context.

[2]Here and in the rest of the proof I will use claims of the form $\phi(\varnothing)$ to abbreviate claims that everything which behaves like the empty set satisfies ϕ i,e. claims of the form $(\exists x)[class(x) \wedge \forall y \neg y \in x \wedge \phi(x)]$.

Now apply Choice to get the $\Diamond_{F,R}$ of an R' which takes the single object in its domain ($\varnothing$) to a single object. By Ignoring (and the fact that the formula $\forall x \forall y(R'(x,y) \to R(x,y)) \land [\forall x F(x) \to \exists!\, y R'(x,y)$ is content restricted to F,R) we can conclude that the above scenario is also $\Diamond_{P,class,\in,F,R}$. Enter the latter $\Diamond$. By simple comprehension we can have $\Diamond_{P,class,\in,R,F,R'}$ P' applies to the single object which R' relates $\varnothing$ to.

Enter this final $\Diamond$ context. Because our biconditionals characterizing R, F and R' are suitably content-restricted, we can import them through all the $\Diamond$s for use in the current $\Diamond_{P,R,F,R'}$ context. Thus we can deduce that $(\exists x)(P(x) \land P'(x) \land (\forall y)[P'(y) \to x = y])$ is true in this $\Diamond_{P,class,\in,R,FR'}$ context.

Leaving this context, we can conclude that $\Diamond_P(\exists x)(P(x) \land P'(x) \land (\forall y)[P'(y) \to x = y])$ by $\Diamond$E. Now this claim is content restricted to P, so we can pull it out of all the various $\Diamond$ contexts (each of which holds fixed the application of P) one by one.

Thus, we can conclude $\vdash (\exists x)P(x) \to \Diamond_P(\exists x(P(x) \land P'(x) \land (\forall y)[P'(y) \to x = y])$, as desired.

Simple Choice for N-tuples $\vdash (\exists \vec{x})R(\vec{x}) \to \Diamond_R(\exists \vec{x}(R'(\vec{x}) \land (\forall \vec{y})[R'(\vec{y}) \to \vec{x} = \vec{y}])$

We can prove all claims of this form by applying the following strategy. First suppose for $\to$I, that $(\exists \vec{x})R(\vec{x})$.

Now apply Possible Powerset a bunch of times (holding fixed R and entering $\Diamond$s after each time) until you have enough layers of sets to have sets corresponding to $\vec{x}$ (as per the usual set theoretic way of associating ordered n-tuples with sets). By simple comprehension, P could apply to exactly those sets coding ntuples $\vec{x}$ such that $\boldsymbol{R}\vec{x}$. Enter this $\Diamond_{R,set1,set2....setn}$ context. By the previous lemma we have $\Diamond_P(\exists x(\boldsymbol{P}(x) \wedge \boldsymbol{P'}(x) \wedge (\forall y)[\boldsymbol{P'}(y) \rightarrow x = y])$. By ignoring we can make this $\Diamond_{P,R,set_1,set_2....set_n}$. Enter the latter $\Diamond$ context. All the facts characterizing the $\boldsymbol{sets_i}$ are suitably content-restricted, so they can be imported. By simple comprehension, it is also logically possible (fixing all the relations mentioned above) that $\boldsymbol{R'}$ applies to exactly single n-tuple $\vec{x}$ coded by the unique set which $\boldsymbol{P'}$ applies to. So, by importing all the previously mentioned facts characterizing $\boldsymbol{R}, \boldsymbol{P}, \boldsymbol{P'}$ and the $\boldsymbol{set_i}$, and then applying a bunch of first order logic we can derive that $(\exists\vec{x}(\boldsymbol{R}(\vec{x}) \wedge \boldsymbol{R'}(\vec{x}) \wedge (\forall\vec{y})[\boldsymbol{R'}(\vec{y}) \rightarrow \vec{x} = \vec{y}])$.

Finally, we can leave the above $\Diamond$ context and conclude that $\Diamond_R(\exists\vec{x}(\boldsymbol{R'}(\vec{x}) \wedge (\forall\vec{y})[\boldsymbol{R'}(\vec{y}) \rightarrow \vec{x} = \vec{y}])$, by In$\Diamond$. Since this formula is content restricted to R, so we can bring it out of all the $\Diamond$ contexts we have entered (all of which hold fixed $\boldsymbol{R}$), just as above.

This gives us $\Diamond_R(\exists\vec{x}(\boldsymbol{R'}(\vec{x}) \wedge (\forall\vec{y})[\boldsymbol{R'}(\vec{y}) \rightarrow \vec{x} = \vec{y}])$, and thus the desired conditional.

11.7 The Wrapping Trick

My final lemma lets us mirror some of the power of quantifying in. [figure this out]

The Wrapping Trick

Let ϕ be a formula which is content restricted to the L structure/so that the sentence $(\forall x)(x \in Ext(\mathcal{L}) \rightarrow \phi(x))$ is a well formed sentence, and content restricted to $\mathcal{L}$.

Then, if we can infer contradiction from the hypothesis that the $\mathcal{L}$ structuere is preserved, while Q picks out a single thing from within the L structure which doesn't satisfy $\phi(x)$ (as below),

i.e. if $\Gamma \vdash \Box_L[(\exists!\, x : Q(x)) \wedge x \in Ext(\mathcal{L})\neg\phi(x) \rightarrow \bot]$

then we can infer that $\phi(x)$ is true of everything within the L structure i.e.

$\Gamma \vdash (\forall x)(x \in Ext(\mathcal{L}) \rightarrow \phi(x))$

Proof. Let ϕ satisfy the condition above. Assume that $\Box_L[(\exists!\, x : Q(x)) \wedge x \in Ext(\mathcal{L})\neg\phi(x) \rightarrow \bot]$.

And suppose, for contradiction, that not $(\forall x)(x \in Ext(\mathcal{L}) \rightarrow \phi(x))$.

Note that by the fact that the above is a well formed sentence ϕ cannot contain any boxes or diamonds in which x occurs as a free variable.

Then by modal comprehension we can (fixing the facts about L) have predicate P which applies to all x in $\mathcal{L}$ such that $\neg\phi(x)$

And by choice we can have Q apply to exactly one thing stisfying P.

So, by multiple definitions we have $\Diamond_L(\exists!\, x : Q(x)) \wedge x \in Ext(\mathcal{L})\neg\phi(x)$.

Enter this $\Diamond$ contenxt.

Now we can import the fact that $\Box_L[(\exists!\, x : Q(x)) \wedge x \in Ext(\mathcal{L})\neg\phi(x) \rightarrow \bot]$ into this context, and hence infer that $(\exists!\, x : Q(x)) \wedge x \in Ext(\mathcal{L})\neg\phi(x) \rightarrow \bot$ and $\bot$

So, stepping out of this context we have $\Diamond_L(\bot)$ and hence $\bot$. This gives the desired contradiction.

Thus we have $(\forall x)(x \in Ext(\mathcal{L}) \rightarrow \phi(x))$

$\Box$

Chapter 12

Stronger Infinity Lemma and the Wrapping Trick

In this chapter I aim to do two things. First, I will show that Infinity Axiom above (together with my other inference rules) above implies the more practically useful, Infinite Well-Ordering Lemma below.

Second, I will introduce the reader to a key argumentative move, 'The Wrapping Trick', which I will use many times in the argument to come.

Recall that the infinity lemma says the following.

Axiom 12.0.1 (Infinity)**.** *It is possible for a two place relation S to apply in the following successor-like way:*

- *The successor of an object is unique* $(\forall x)(\forall y)(\forall y')[S(x, y) \wedge S(x, y') \rightarrow y = y']$
- *successor is one-to-one* $(\forall x)(\forall y)(\forall x')(S(x, y) \wedge S(x', y) \rightarrow x = x')$

- *there is a unique object that has a successor and isn't the successor of anything* $(\exists!\, x)(\exists y)\,(S(x,y) \wedge (\forall y)\,[\neg S(y,x)])$
- *everything that is a successor has a successor* $(\forall x)[(\exists y)S(y,x) \rightarrow (\exists z)S(x,z)]$
- S *is anti-reflexive:* $(\forall x)(\forall y)[S(x,y) \rightarrow \neg S(y,x)]$

I will show that this claim that we could have have an infinite sequence of objects related by some relation in a successor-like way implies the more useful (for our purposes) claim that we could have an infinite well ordering.

Proposition 12.0.2 (Infinite Well-Ordering Lemma)**.** *It is logically possible for there to be a non-empty well ordering with no maximal element. That is, it is logically possible that*

- $<$ *well-orders the objects satisfying* W *(as per definition)*
- $(\exists x)(W(x))$
- $(\forall x \mid W(x))(\exists y \mid W(y))\,(x < y)$

The idea behind my proof will be that, given any logically possible scenario where some two place relation S applies as per the infinity axiom above, it is logically possible (while preserving these facts about S) to have $W, \leq$ apply so that the following conditions are satisfied (and that this suffices to ensure that $W, \leq$ pick out a non-empty well ordering with no maximal element)

- W applies to exactly the objects that belong to every successor-closed set containing the 0 object

- $x \leq y$ iff every successor-closed set containing x contains y

Structurally, I will proceed by considering an arbitrary logically possible scenario witnessing the truth of the infinity axiom above. I will first argue that (holding fixed the facts about S in this scenario) it is logically possible for $\boldsymbol{W}, \leq$ to apply a as above. I will then check that the itemized claims above must hold of $\boldsymbol{W}, \leq$ in this scenario, so that they form a non-empty well ordering with no maximal element, as desired. And I will conclude by using the Diamond Collapsing lemma to infer from this $\Diamond\Diamond_{\boldsymbol{S}}(\boldsymbol{W}, \leq$ is a non-empty well ordering with no maximal element) claim to the desired conclusion that $\Diamond(\boldsymbol{W}, \leq$ is a non-empty well ordering with no maximal element).

So let us begin.

By the infinity axiom [ref] it is logically possible to have an injective function $\boldsymbol{S}(x, y)$ defined on $\mathrm{Ext}(\boldsymbol{S})$ with a unique element 0 which isn't a successor. Let $\mathcal{S}(\boldsymbol{S})$ denote the conjunction of these claims. Then we have $\Diamond\mathcal{S}(\boldsymbol{S})$.

Now let if we enter this $\Diamond$ context, we can use the Multiple Definitions Lemma to argue for the simulatnious satisfiability of the following chain of cumulative stipulations:

- $(\Diamond_{\boldsymbol{S}})$ D applies so that $(\forall x)[\boldsymbol{D}(x) \leftrightarrow \boldsymbol{Ext}(x, \boldsymbol{S})]$ (by Simple Comprehension), and in this scenario...

- $(\Diamond_{\boldsymbol{S},\boldsymbol{D}})$ $\boldsymbol{C}, \in$ apply so that $\mathscr{C}(\boldsymbol{C}, \in, \boldsymbol{D})$ (by Possible Powerset)

- $(\Diamond_{\boldsymbol{S},\boldsymbol{D},\boldsymbol{C},\in})$ $\boldsymbol{SC}$ applies so that $\boldsymbol{SC}(x)$ iff $\boldsymbol{C}(x) \wedge (\forall z)(\forall z')(\in (z, x) \wedge \boldsymbol{S}(z, z') \rightarrow \in (z', x))$ [In such cases I will say that x is 'successor closed']

- ($\Diamond_{S,D,C,\in,SC}$) $\leq$ applies so that $x \leq y$ iff every S-closed set containing x contains y , i.e., $(\forall x)(\forall y)(x \leq y \leftrightarrow (\forall k)[x \in k \wedge SC(k) \rightarrow y \in k])$ (by Simple Comprehension)

- ($\Diamond_{S,D,C,\in,SC,\leq,}$) W applies to exactly the objects that belong to every S-closed set containing the 0 object $(\forall x)[W(x) \leftrightarrow (\exists z)(D(z) \wedge (\forall y)\neg S(y,z) \wedge (\forall k)[SC(k) \wedge y \in k \rightarrow x \in k])]$ (by Simple Comprehension)

So, by the multiple defintions lemma we know that this possible scenario in which $\mathcal{S}(S)$ is also one in which $\Diamond_S\Phi$ in this context (where Φ is the conjunction of the five bullet pointed claims as above).

Thus (returning to our original context) we can infer $\Diamond\mathcal{S}(S)\wedge\Diamond_S(\Phi)$. As $\mathcal{S}(S)$ is implicitly content restricted to S, we can use Diamond Collapsing to infer that we could simultaniously have $\mathcal{S}(S)$ and $D, C, \in, W, \leq$ applying in the way specified above i.e. $\Diamond(\mathcal{S}(S) \wedge \Phi)$.

This completes the construction of $W, \leq$, our *intended* non-empty well ordering with no maximal element. We must now check that this logically possible scenario really behaves as advertised (i.e., that, because $\mathcal{S}(S) \wedge \Phi$, $W, \leq$ must really apply to out a non-empty well ordering with no maximal element).

W is non-empty: If $\mathcal{S}(S) \wedge \Phi$ then $\exists x W(x)$.

Clearly W is non-empty, because 0 is an element of every successor closed class containing 0, and hence an element of 0.

Before checking the other conditions I will prove the following following lemma, which will help us show that $W, \leq$ satisfies the anti-reflexivity requirement above. Proving it also provides a nice opportunity to introduce an argumentative technique Wrapping Trick mentioned above, which lets us mimic some of the power of quantifying in (and will be frequently reused in the remainder of this book).

Lemma 12.0.3. *[Successor Lemma] If $\mathcal{S}(S) \wedge \Phi$ then*$(\forall x)[W(x) \rightarrow \neg S(x) \leq x]$[1]

Intuitively, this lemma says if $\mathcal{S}(S) \wedge \Phi$ then nothing satisfying W is $\leq$ its successor.

Proof. Consider an arbitrary situation in which $\mathcal{S}(S) \wedge \Phi$. By Simple Comprehension, it would be logically possible[2] for a property G (for 'good') to apply to all those x such that $W(x)$ which conformed to the above principle, i.e. those such that $\neg S(x) \leq x$ [3]

So, by our characterization of the layer of classes $C, \in$ over the objects satisfying D, we have added, we can infer that there's already class g which collects all these x such that $W(x) \wedge \neg S(x) \leq x$. (Note that all objects satisfying W satisfy must satisfy D, because the class that contains all of D is successor-closed and contains 0).

If we can show that the class g of 'good' objects contains 0 and is closed under our 'successor' relation S, then it follows (by our characterization of W) that every object satisfying W belongs to g, so no object satisfying W

[1] $(\forall x)(\forall y)[W(x) \wedge S(x, y) \rightarrow \neg x \leq y]$
[2] holding fixed the facts about $S, D, C, \in, SC, \leq, W$
[3] That is $\Diamond_{S,D,C,\in,SC,\leq,W}(\forall x)(G(x) \leftrightarrow W(x) \wedge \neg S(x) \leq x)$

is $\leq$ its successor, and the lemma is true. Thus, it suffices for us to prove the following two claims.

The 0 object belongs to g, the class associated with G: Clearly, $\boldsymbol{W}(0)$. We must show that $\neg\boldsymbol{S}(0) \leq 0$. By our characterization of $\leq$, this means showing that there is a class which is successor-closed and contains $\boldsymbol{S}(0)$, but which does not contain 0. I will argue that $\{x|\boldsymbol{W}(x) \wedge \neg x = 0\}$ (i.e., the class of $\boldsymbol{W}$s which are not the 0 object) does the trick.

This class exists, by our characterization of the layer of classes over the objects satisfying $\boldsymbol{D}$ (and the fact that everything that satisfies $\boldsymbol{W}$ satisfies $\boldsymbol{D}$ as noted above). Obviously it does not contain 0.

It does contain $\boldsymbol{S}(0)$. For we know that $\boldsymbol{S}(0)$ satisfies $\boldsymbol{W}$, because clearly every successor closed class containing 0 contains the successor of 0. And we know that $\neg\boldsymbol{S}(0) = 0$, because 0 is not the successor of anything.

And it is successor-closed, because the objects satisfying $\boldsymbol{W}$ are closed under successor, and 0 is not the successor of anything, so $\{x|\boldsymbol{W}(x)\} - 0$ must be closed under successor as well.

This proves our base case $\neg\boldsymbol{S}(0) \leq 0$.

g is closed under successor: It suffices to prove that if x satisfies $\boldsymbol{G}$ then $\boldsymbol{S}(x)$ satisfies $\boldsymbol{G}$, i.e., $(\forall x)[\boldsymbol{W}(x) \wedge \neg x \leq \boldsymbol{S}(x) \rightarrow \boldsymbol{W}(\boldsymbol{S}(x)) \wedge \neg\boldsymbol{S}(x) \leq \boldsymbol{S}(\boldsymbol{S}(x)))]$.

At first glance, we'd like use $\forall$-introduction to consider an arbitrary good x (i.e., an x such that $\boldsymbol{W}(x) \wedge \neg x \leq \boldsymbol{S}(x)$) and then show that $\boldsymbol{S}(x)$ must be good as well. If we knew that there was a successor closed-class $\boldsymbol{c}$ in which S(x) is the least element, then we could show $\boldsymbol{S}(\boldsymbol{S}(x))$ is in this class and then construct another successor closed class $\boldsymbol{c}'$ containing S(S(x)) but not S(x), thus showing that $\neg\boldsymbol{S}(\boldsymbol{S}(x)) \leq \boldsymbol{S}(x)$.

It might seem that we could define the needed c by comprehension as containing those elements y such that $W(y) \wedge S(x) \leq y \wedge \neg y = S(x)$.[4] However, our Simple and Modal Comprehension principles only applies to complete sentences, not formulae with free variables.

Instead we must take a slightly more complicated approach, employing a generally useful technique which I will call **The Wrapping Trick**. This principle allows us to mimic certain kinds of reasoning that might seem to require quantifying in, by using choice and proof by contradiction.

The Wrapping Trick

To prove a 'universal claim' about all objects within some structure $\mathcal{L} = R_1 ... R_n$ (i.e., a claim of the form $\forall x \phi(x)$ which is content restricted to this structure) via the Wrapping Trick, we suppose, for contradiction, that this claim is false. Then we use choice deduce the logical possibility that (holding fixed the facts about $\mathcal{L}$) some new relation $Q(\cdot)$ applies to a unique object x in the $\mathcal{L}$ structure such that x such that $\neg\phi(x)$ (in this case, Q applies to a unique x, such that x is good but $S(x)$ is not). In this logically possible scenario, we *can* define c as above, using Simple Comprehension over a sentence using Q rather than formula using x. Thus we can complete the above argument that $\neg S(S(x)) \leq S(x)$ (where x is taken to be the unique object satisfying Q), so $S(x)$ is good. This contradicts our initial characterization Q, yielding the conclusion that $\perp$ holds within this $\Diamond$ context. Finally we can infer from $\Diamond\perp$ to $\perp$, securing the contradiction

[4] i.e., it might seem that we could use simple comprehnsion to deduce that it would be that a property P could apply to exactly these objects, and then use our characterization of the layer of classes to infer that such a c exists.

desired.

Now let's fill in the details. The universal claim we need to prove is $(\forall x)[W(x) \wedge \neg x \leq S(x) \rightarrow W(S(x)) \wedge \neg S(x) \leq S(S(x)))]$. So suppose, for contradiction, that it is false. Then there is a counterexample to it, and by Simplified Choice it is logically possible (holding fixed $S, W, \leq, C, \in, D, SC$ and any other relations we desire), that an otherwise-unused predicate Q applies to a unique object x and this object is a counterexample, i.e., $\neg[W(x) \wedge \neg x \leq S(x) \rightarrow W(S(x)) \wedge \neg S(x) \leq S(S(x)))]$.

Enter this logically possible scenario. We know that there is a unique object satisfying Q. Call it x. We also know that $\neg[W(x) \wedge \neg x \leq S(x) \rightarrow W(S(x)) \wedge \neg S(x) \leq S(S(x)))]$. .

Thus we can deduce that $W(x) \wedge \neg S(x) \leq x$. We will derive $\bot$ by proving that $W(S(x)) \wedge \neg S(S(x)) \leq S(x)$. It is easy to see that $S(x)$ exists and satisfies W.[5] As previously mentioned, we will prove that $\neg S(S(x)) \leq S(x)$ by showing that there's a successor-closed class k which contains $S(S(x))$ but not $S(x)$. By Simple Comprehension and our characterization of the classes we can show that there is a class $\{y | W(y) \wedge S(x) \leq y \wedge \neg y = S(x)\}$, because x is the unique object satisfying Q, so this class can be characterized as $\{y | W(y) \wedge (\forall x)[Q(x) \rightarrow S(x) \leq y \wedge \neg y = S(x)]\}$.[6]

[5] $S(x)$ exists by the fact that everything satisfying W satisfies D (noted above) and the fact that everything satisfyind D has a successor. Since $W(x)$, we know that every SC class containing 0 contains x. This implies that every SC class containing 0 also contains $S(x)$, so $W(S(x))$.

[6] First we use simple comprehension to say that a predicate $P()$ could $\Diamond_{W,\leq,S,D,class,\in,Q}$ apply to exactly the objects with the property above. Then we note that this is a situation in which by our characterization of the classes must remain true (because it is content restricted to $D, class, in$). Thus we have $\Box_{D,class,\in}$(if P only applies to objects satisfying D then there is a class corresponding to the extension of P). Now we apply $\Box$E to get the truth of the conditional claim inside. We can fairly straightforwardly deduce that P only

This class contains $S(S(x))$ because $S(x) \leq S(S(x))$, and nothing is its own successor (by antireflexivity), so $\neg S(S(x)) = S(x)$. But this class is also closed under successor. For, consider any y such that $S(x) \leq y \land \neg y = S(x)$. We know that $S(x) \leq S(y)$, since if every successor-closed set containing $S(x)$ contains y, then every successor closed set for $S(x)$ contains $S(y)$. But we can also show that $\neg S(y) = S(x)$, since successor is 1-1 and we know that $\neg y = x$ because $y \geq S(x)$ but (by inductive hypothesis) not $x \geq S(x)$.

Thus, there is successor closed class k which contains $S(S(x))$ but not contain S(x), and hence $\neg S(x) \leq S(S(x))$, as desired. This contradicts our assumption that x (the unique object satisfying C) is a counterexample to the lemma. So we have $\bot$ inside this modal context.

Finally, we can pull this proof of contradiction back to our original scenario. Leaving the above $\Diamond$ context, we have $\Diamond_{R_1 \ldots R_n} \bot$ and hence $\bot$, which completes our proof by contradiction.

$\square$

In the pages that follow, we will frequently use the above method of introducing a new predicate Q which applies to a single object with a given property (here it was being a counterexample to the claim that the successor of every good object is good) and then reasoning using this relation (inside the logical possibility context introduced to define it) to refer to a witness with this property. I will call this method the **Wrapping Trick**. Note

applies to objects satisfying D. So we can infer that there is a class which behaves like the extension of P, Finally, we can deduce that that there is a class containing exactly the objects which satisfy $W(y) \land (\forall x)[Q(x) \rightarrow S(x) \leq y \land \neg y = S(x)]$. Because the latter claim is content restricted to $W, \leq, S, D, class, \in, Q$, we can reason from its truth in this logically possible scenario to its truth in the scenario previously under consideration.

that the context in which the new predicate Q is introduced will always be relativized to all other relations mentioned so far in the proof, allowing us to move needed results into this $\Diamond$ context (and pull our conclusion out of it).

With the above lemma in hand, let us now turn to the mainproof. To establish that $W, \leq$ is a well ordering, we must check four things:

reflexivity: $(\forall x)(W(x) \rightarrow x \leq x)$. Consider an arbitrary x such that $W(x)$. Obviously, every set which contains x and is closed under successor contains x. So $x \leq x$.

transitivity: $(\forall x)(\forall y)(\forall z)(W(x) \wedge W(y) \wedge W(z) \wedge x \leq y \wedge y \leq z \rightarrow x \leq z)$. Consider arbitrary x, y and z such that $W(x) \wedge W(y) \wedge W(z) \wedge x \leq y \wedge y \leq z$. Clearly if every successor-closed class that contains x contains y, and every successor-closed class that contains y contains z, then every successor-closed class that contains x contains z. So $x \leq y$.

comprability. $(\forall x)(\forall y)[W(x) \wedge W(y) \rightarrow x \leq y \vee y \leq x]$.

Consider the property of of being an x such that $(\forall y)(W(y) \rightarrow x \leq y \vee y \leq x)$. Just as in the lemma above, we know that there is a class g of all the Ws which have this 'good' property[7] and I will argue that this class is successor-closed and contains 0. From this it follows that this class contains all objects satisfying W (so the comprability condition above is satisfied).

0 belongs to this class: It is immediate from our characterization of W

[7]This follows from our characterization of the layer of classes over the objects satisfying D (and the fact that every object satisfying W satisfies D noted above)

that $(\forall y)((W(y) \to 0 \leq y)$.

If x belongs to this class then S(x) does: [discuss][We can use the Wrapping Trick (twice) to reproduce the following reasoning about a free variables x and y, using predicates Q and Q'.] Suppose, for contradiction, that some x satisfies the conditions for being 'good' but $S(x)$ does not. Then we have $(\forall y)(x \leq y \vee y \leq x)$ but also $(\exists y)\neg(S(x) \leq y \vee y \leq S(x))$. Now consider any y witnessing the latter fact. We know that $x \leq y \vee y \leq x$, but:

- We cannot have $y \leq x$. For, if every successor-closed class containing y contains x, then each such class also contains $S(x)$, so $y \leq S(x)$. Contradiction.

- We cannot have $x \leq y$. For, by hypothesis, $\neg S(x) \leq y$, so there's a successor-closed class containing $S(x)$ which does not contain y. Then there's a class which contains these objects and x (by an application of simple comprehension and our characterization of the layer of classes).[8] But this is a successor-closed class containing x which does not contain y, so $\neg x \leq y$. Contradiction.

least element: $\Box_{W,\leq}[(\exists x)(K(x) \wedge W(x)) \to (\exists x')(K(x') \wedge W(x') \wedge (\forall y)[K(y) \wedge W(y) \to x' \leq y])]$

This condition asserts that (restricting our attention to the objects satisfying W) if a predicate K holds for some x satisfying W, then there is a $\leq$-least x satisfying W and K.

Suppose, for contradiction, that it were $\Diamond_{W,\leq}$ to have $(\exists x)(K(x) \wedge W(x))$

[8] Remember that the Wrapping Trick has us consider a situation where a predicate Q applies to our putative counterexample x and Q' to some choice of y.

but not $(\exists x')(K(x') \wedge W(x') \wedge (\forall y)[K(y) \wedge W(y) \rightarrow x' \leq y])]$. Informally, this says that some object that satisfies W also satisfies K but there is no $\leq$-*least* such object, i.e., for every x satisfying both K and W, there is a y satisfying K and W such that $\neg x \leq y$. Call this latter claim the No Least Element Assumption, and note that, by first order logic, it can be rewritten as follows: $(\forall x')(K(x') \wedge W(x') \rightarrow (\exists y)[K(y) \wedge W(y) \wedge \neg x' \leq y])]$.

By Ignoring we can deduce that $\Diamond_{class,\in,W,\leq,D}((\exists x)(K(x) \wedge W(x))$ and No Least Element). Now I will consider this supposedly logically possible scenario, and derive a contradiction by showing that $\neg(\exists x)(K(x) \wedge W(x))$.

As usual, we begin by noting that there is a class g of 'good' objects satisfying W, such that nothing $\leq$ them satisfies K, $\{x | W(x) \wedge \forall y \neg(K(y) \wedge y \leq x)\}$.[9] I will show that this class is successor-closed and contains 0. From this it follows *all* objects satisfying W belong to this class, so nothing satisfies both K and W. This establishes the desired conclusion that $\neg(\exists x)(K(x) \wedge W(x))$.

0 belongs to this class: 0 is the only thing ≤ 0, so it suffices to check that $\neg K(0)$. But we know this is true, because the No Least Element Assumption above requires that if K(0) then there is a y such that $W(y) \wedge K(y) \wedge \neg 0 \leq y$. And there can be no such y because, by our characterization of W everything that satisfies W is ≥ 0.

This class is successor-closed: [Just as in the previous proof, we can use two nested applications of the Wrapping Trick to reconstruct the following argument, which treats x and y as free variables.] Suppose, for contradiction,

[9]This follows our characterization of the layer of classes over D and the fact that all Ws are Ds (which remains true in this context because they are content-restricted to $class, \in, W, \leq, D$).

that some x belongs to this class (so nothing ≤ x satisfies K) while its successor does not (so there is a y such that $y \leq S(x) \wedge K(y)$). [discuss][Consider some such x via one applicaiton of the Wrapping Trick. Then consider a witnessing y for this choiceof x via another, nested, application of the Wrapping Trick.]

First we can deduce from the fact that $y \leq S(x) \wedge K(y)$ that $y = S(x)$. We know that everything $\leq x$ doesn't satisfy K. So $\neg y \leq x$, hence there's a successor-closed class c containing y but not x. By simple comprehension[10], there is also another class $c' = c - \{S(x)\}$. Our original class c cannot contain any predecessor for $S(x)$, since it does not contain x and S is a 1-1 function. Thus unless $y = S(x)$, removing $S(x)$ from this original class c will leave something which contains y and is still closed under successor, but does not contain $S(x)$ (contradicting $y \leq S(x)$). Thus we have $y = S(x)$ and hence $S(x)$ satisfies K and W.

Now the No Least Element Assumption above requires that there is a z such that $\neg S(x) \leq z$ and $W(z) \wedge K(z)$. So there's a successor-closed class c containing $S(x)$ but not this z. And there is a successor-closed class $c' = c + \{x\}$ which adds x to c. We know that c' still doesn't contain z (because K(z) but not K(x) so $\neg z = x$). Thus we have $\neg x \leq z$. By the comprability property just proved above we have $z \leq x$. So we have $(\exists z)(z \leq x \wedge K(x))$, contradicting our initial choice of x.

This completes the above argument that every object satisfying W belongs to the class above, so that $\neg(\exists x)W(x) \wedge K(x)$ and we can derive $\perp$ within the $\Diamond$ context under consideration.

[10] And the Wrapping Trick

However, as discussed previously $\bot$ is content-restricted any list of relations, so we can infer from $\Diamond\bot$ to $\bot$. Thus we have $\neg\Diamond_{W,\leq}\neg(\exists x)[K(x) \wedge W(x) \to (\exists x')[K(x') \wedge W(x) \wedge (\forall y)(K(y) \to x \leq y)]$, as desired.

everything is either 0 or a successor: $(\forall x)[W(x) \to x = 0 \vee (\exists y)(S(y) = x)]$

[We can use the Wrapping trick above to reconstruct the following argument.] Suppose there is an a such that $W(a)$ but neither $a = 0$ nor $(\exists a_0)S(a_0) = a$. We can show that $\neg W(a)$ (and hence get a contradiction) by deducing the existence of a successor-closed class which contains 0 but does not contain a as follows. By our characterization of the classes over D, there is a class of all objects satisfying D. By the fact that everything in Ext(S) satisfies D and 0 has a successor, this is a successor-closed class which contains 0. There is also a class c' formed by removing only a from this class. By the assumption that a is not 0 and not the successor of anything, this class will also be successor closed and contain 0.

antisymmetry: $(\forall x)(\forall y)(x \leq y \wedge y \leq x \to x = y)$

Proof. Suppose, for contradiction, that $(\exists x)(\exists y)(W(x) \wedge W(y)x \leq y \wedge y \leq x \wedge \neg x = y)$. Using the least element condition already established, we can quickly show that there is $\leq$-least object a such that $(\exists y)(W(a) \wedge W(y) \wedge a \leq y \wedge y \leq a \wedge \neg a = y)$.[11] By the same reasoning, we can further deduce that

[11]Specifically, by simple comprehension, a property K could apply to exactly the Ws with the property above. Consider this $\Diamond$ scenario. The least element condition must remain true. So there is an $\leq$-least object a such that $(\exists y)(W(a) \wedge W(y) \wedge a \leq y \wedge y \leq a \wedge \neg a = y)$ in this scenario. But the latter claim is content restricted to $W, \leq$ so it must be true in our original scenario as well.

there is an $\leq$-least b, which satisfies $W(a) \wedge W(b) \wedge a \leq b \wedge b \leq a \wedge \neg a = b$ for this $\leq$-least a.

I will argue by dilemma. First note that because $W(a)$, we know that either $a = 0$ or $(\exists a_0)(S(a_0) = a)$ by the fact just proved above.

First consider the case where $a = 0$.

I will show that $\neg b \leq 0$, contradicting our choice of b so that $b \leq a$. By simple comprehension and our characterization of the classes there is a $c = \{x|W(x) \wedge x \geq b\}$. This class is successor-closed, since it is immediate from unpacking definitions that the successor of anything $\geq b$ is $\geq b$.

By another application of simple comprehension(as above) we know there is a class $c' = c - 0$. We can show this is also a successor-closed class, and that it containing b but not 0 (so that $\neg b \leq 0$ as desires) as follows. We know that it still contains b, because c does, 0 is the only thing that got removed from c, and $\neg a = b$ so (given our current assumption that a=0) $\neg 0 = b$. It is successor-closed because c is, and the only item which is removed (0) is not the successor of anything.

Now consider the case where $(\exists a_0)(S(a_0) = a)$. By Lemma 12.0.3 above we have $\neg S(a_0) \leq a_0$, hence $\neg a \leq a_0$. So by our characterization of a as the least thing with the property above, we know that a_0 does not have this property. So either $\neg a_0 \leq b \vee \neg b \leq a_0$ or $a_0 = b$.

- if $\neg a_0 \leq b$ then there's a successor-closed class containing a_0 which doesn't contain b, hence a successor-closed class containing a that doesn't contain b. So $\neg a_0 \leq b$. Contradiction.

- if $a_0 = b$, then since $b \leq a$ we have $a_0 \leq a$. But this is impossible by the Lemma, just mentioned.

- if $\neg b \leq a_0$, then there's a successor-closed class c containing b that doesn't contain a_0. Now it suffices to show that we can turn this into a successor-closed class containing b but not a. For, from this it follows that $\neg b \leq a$, which contradicts our choice of a.

  By our characterization of the layer of classes (and simple comprehension), we know that there is a class $c' = c - \{a\}$. The resulting class still contains b because $\neg b = a$ by the argument above. And it remains closed under successor because the single item we have removed (a) cannot be the successor of anything in c or c' (since successor is 1-1 and a is the successor of a_0 which is not in c).

□

Finally, it is easy to check that $W, \geq$ is an *infinite* well ordering in the sense characterized above: that it contains an object and contains something $\geq$-larger than every object which it contains.

infinite well ordering:

Proof. W is a well-ordering by the argument above. We know that W applies to something because clearly the 0 object satisfies W. And we can show that W contains some object strictly $\geq$-larger than every object which it contains, as follows (using the Wrapping Trick as usual).

Suppose some x satisfies W. Then $S(x)$ exists and satisfies W, as noted above.[12] Clearly every successor-closed class containing x contains $S(x)$, so

[12] Every successor-closed class containing 0 contains x. So each of these classes also

$S(x) \geq x$. By the antireflexivity of S, we have $\neg S(x) = x$. So $S(x)$ is strictly $\geq$-larger than x.

□

contains $S(x)$. Thus $S(x)$ also satisfies W.

Chapter 13

Vindication of FOL Inference

We are now in a position to begin the main task of this monograph, by justifying and explaining mathematicians' ordinary use of first order logic in set theory, from a potentialist point of view. Note that my potentialist translations of a set theoretic sentence have a very different logical form from the original. Thus, it is not immediately obvious that whenever a set theoretic sentence β is a first order logical consequence of α, then $t(\beta)$ is a genuine logical consequence of $t(\alpha)$. And it is not obvious that set theorists' willingness to move from a sentence that looks like it asserts α to one that looks like it asserts β is justified.

In this section I will show that every first-order logical argument in the language of set theory can be transformed into an argument in the deduction system described above, which takes us from the translation of the premises for this argument to the translation of its conclusion. In particular, I will establish the following claim.

Proposition 13.0.1. *If $\gamma_1, \ldots, \gamma_m, \theta$ are sentences in the language of set theory and $\gamma_1 \ldots \gamma_m \vdash_{FOL} \theta$ then $t(\gamma_1) \ldots t(\gamma_m) \vdash (\theta)$ (where $\vdash_{FOL}$ represents provability in first order logic and $\vdash$ represents provability in the formal system just introduced).*

13.1 Translation Lemma

A key tool in making this argument will be the following Translation Lemma.

Translation Lemma: If $v_1 \ldots v_k$ are the only variables free in a set theoretic formula ϕ, then $\vdash V_n \geq V_m \wedge f_n = f_m(v_1) \wedge ...f_n(v_k) = f_m(v_k) \rightarrow (t_n(\phi) \leftrightarrow t_m(\phi))$.

Remember that $V_n \geq V_m$ means that V_n extends V_m when considered merely as a hierarchy of sets (unlike $\vdash \vec{V}_n \geq \vec{V}_m$ it does not require any agreement between the assignment functions f_n and f_m).

Intuitively, the Translation Lemma says that the way V_n, f_n assigns the free variables in a set theoretic formula ϕ completely determines whether $t_n(\phi)$ is true. Specifically, the truth-value of $t_n(\phi)$ which talks about how V_n, f_n can be extended must agree with that of any $t_m(\phi)$ which talks about the same assignment of all the free variables in ϕ as considered within a larger hierarchy of sets V_m, f_n extending V_n, f_n.]

Note: Hellman proves something analogous to this lemma in [3], assuming the axiom of inaccessibles. [add quote]

Proof. I will prove this claim by induction on complexity of formulas.

This principle is fairly obviously true for atomic sentences, which all take the form $x = y$ or $x \in y$. For, if $f_n(x) = f_m(x) \wedge f_n(y) = f_m(y) \rightarrow$

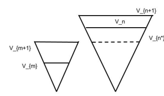

Figure 13.1: One possible relation relationship between V_m, V_{m+1}, V_n and the V_*, V_{n+1} we will construct.

$(f_n(x) = f_n(y) \leftrightarrow f_m(x) = f_m(y))$. Similarly, we get $f_n(x) = f_m(x) \wedge f_n(y) = f_m(y) \rightarrow (f_n(x) \in_n f_n(y) \leftrightarrow f_m(x) \in_m f_m(y))$ by $V_n \geq_{set} V_m$. It is also obvious that applying the truth-functional connectives $\wedge$, $\vee$ and $\neg$ to formulae which have this property always yields new formulas which has this property.

Assume $f_n(v_1) = f_m(v_1) \wedge ...$ for all variables $v_1 \dots v_k$ free in $(\forall x)\phi(x)$. Assuming that the relevant theorem holds for ϕ, we will show that it holds for $(\forall x)\phi(x)$ as well.

Suppose, for contradiction, that we had $t_n((\forall x)\phi(x))$, but not $t_m((\forall x)\phi(x))$. Then we have $\Box_{Vn}[V_{n+1} \geq_{\mathbf{x}} V_n \rightarrow t_{n+1}(\phi)]$. But we also have $\neg t_m((\forall x)\phi(x)) = \Diamond_{Vm}[V_{m+1} \geq_{\mathbf{x}} V_m \wedge \neg t_{m+1}(\phi)]$.

Our strategy will be to build (holding fixed V_n, V_m) a $V_{n+1} \geq_{\mathbf{x}} V_n$ which mimics the assignment of $f_{m+1}(v)$ which makes $t_{m+1}(\phi)$ come out false, and then show that this scenario must be one in which $\neg t_{n+1}(\phi)$ (contrary to the $\Box_{Vn}$ claim above, which can be imported into this $\Diamond_{V_n, V_m}$ context).

By the Hierarchy Extending Lemma (proved in appendix C.5) we can have V_{n+1} extending V_n, containing an isomorphic copy, V_{n*}, of V_{m+1}.

Lemma 13.1.1 (Hierarchy Extending Lemma)**.** *If $\mathscr{V}(V_a) \wedge \mathscr{V}(V_b)$ then it's possible (holding fixed V_a, V_b) that there is a V_{a+} extending V_a and Z' func-*

tioning like an isomorphism from V_b to an intial segment of V_{a+}

This isomorphism between sets (and the unique isomorphism between copies of the numbers) naturally induces an isomorphism for f, so now we have V_n*, f_n* isomorphic to V_{m+1}, f_{m+1}. Then by the Isomorphism Lemma (proved in appendix B.2), we can infer from the falsehood of $t_{m+1}(\phi)$ to the falsehood of $t_{n*}(\phi)$ (where $t_{n**}(\phi)$ is the version of $t_n(\phi)$ which talks about V_n*, f_n* rather than V_n, f_n[1]).

Ismorphism Lemma *If ϕ is a formula employing only relation symbols $R_1 \ldots R_n$ and quantifiers restricted to objects related by $R_1 \ldots R_n$*[2] *outside of all $\Box$s and $\Diamond$s, ϕ' is the result of replacing each R_i in ϕ with a corresponding R'_i (which does not occur anywhere in ϕ), and $x_1, ...x_n$ are all the free variables in ϕ then:*

$\vdash \langle R_1, \ldots R_m \rangle \cong_Z \langle R'_1, \ldots R'_m \rangle \rightarrow (\forall x_1)...(\forall x_n)...(\forall x'_1)...(\forall x'_n)[Z(x_1, x'_1) \wedge ...Z(x_n, x'_n) \rightarrow (\phi(x_1...x_n) \leftrightarrow \phi'(x'_1...x'_n))]$

By Simple Comprehension (and the Multiple Definitions Lemma proved on 11.5), we can have f_{n+1} such that $\vec{V_{n+1}} \geq_x \vec{V_n}$ and $f_{n+1}(x) = f_{n*}(x)$. Now we want to show that f_{n+1} and f_{n*} will agree on all variables free in ϕ, and then deploy (a version of) the inductive hypothesis to infer from $\neg t_*(\phi)$ to $\neg t_{n+1}(\phi)$.

[Directly, the inductive hypothesis only gives us $\vdash V_{n+1} \geq V_n \wedge f_{n+1} = f_n(v_1) \wedge ...f_{n+1}(v_k) = f_n(v_k) \rightarrow (t_{n+1}(\psi) \leftrightarrow t_n(\psi))$. But applying $\Box$I, lets us deduce that this claim is logically necessary. Then applying $\Box$ Relabling

[1]That is, t_{n*} is just like t_n, but replaces all appeals to set_{n+1} with $set_{n+1}*$ and so forth.

[2]i.e., only quantifiers in clauses of the form $(\forall x)(x \in Ext(\hat{R}) \rightarrow ...)$ and $(\exists x)(x \in Ext(\hat{R}) \wedge ...)$, where $\hat{R}$ is a collection of one or more relations from within $R_1 \ldots R_n$, an Ext(R) is as defined as per 7.2.1.

(proved on **??**) lets us derive that the corresponding claim with all instances of f_n replaced by f_{n*} (note that t_{n+1} makes no mention of V_n or f_n) is logically necessary, hence actually true. Thus we have $\vdash V_{n+1} \geq V_{n*} \wedge f_{n+1} = f_{n*}(v_1) \wedge ... f_{n+1}(v_k) = f_{n*}(v_k) \rightarrow (t_{n+1}(\psi) \leftrightarrow t_{n*}(\psi))$, as desired.]

Now it remains to prove the antecedent of the claim above. Clearly f_{n+1} and f_{n*} agree on x. But we can also show that they agree on all the other variables free in ϕ, since for all variables v free in $(\forall x)\phi(x)$, $f_{n+1}(v) = f_n(v) = f_m(v) = f_{m+1}(v) = f_{n*}$, with the last equivalence holding because V_{n*}, f_{n*} was constructed to be isomorphic to f_{m+1}.[3] We also know that $V_{n+1} \geq V_{n*}$, by our construction of V_{n+1}.

Thus, by (our suitably massaged version of) the inductive hypothesis we know that ϕ is true on V_{n*}, f_{n*} iff it is true V_{n+1}, f_{n+1}. Since ϕ is false on V_{n*}, it must be false on V_{n+1} as well. Thus, contrary to the $\Box_{V_n}$ claim above, we have a logically possible scenario (holding fixed V_n) in which $V_{n+1} \geq_{\mathbf{x}} V_n \wedge \neg t_{n+1}(\phi)$. So $\perp$.

The same argument with m and n swapped shows that we can't have $t_m((\forall x)\phi(x))$ be true while $t_n((\forall x)\phi(x))$ is false. Thus we have $t_n((\forall x)\phi(x)) \leftrightarrow$

[3] To spell this argument out rigorously, consider some possible isomorphisms relating $V_n, V_{n+1}, V_m, V_{m+1}$ and V_{n*}. By the fact that $V_n \geq_{set} V_m \vee V_m \geq_{set} V_n$, we know that the identity map $i_{n,m}$ behaves like [alt: it would be possible to have an I which behaves like an identity map, and this would constitute an...] an isomorphism from V_n (or an initial segment of it) to V_m. Consider what happens when we compose this map with the isomorphism z from V_{m+1}, f_{m+1} to V_{n*}, f_{n*}. What we get is an isomorphism from initial segment of V_{n+1} to an initial segment of V_{n+1}. By the Isomorphism Agreement Lemma, we know that this map cannot disagree with the identity isomorphism from V_n to V_n.

Now consider an aribitrary variable v which is not x and occurs in ϕ. It suffices to show that the composite function $z(i_{n,m}(f_n(v))) = f_{n*}(v)$, because then the fact that this composite function behaves like the identity gives us $f_{n*}(v) = f_n(v) = f_{n+1}(v)$.

By hypothesis we have $f_n(v) = f_m(v)$. By the fact that $V_{m+1} \geq_x V_m$, we have $f_m(v) = f_{m+1}(v)$, and hence $i_{n,m}(f_n(v)) = f_m(v) = f_{m+1}(v)$. By the fact that z as an isomorphism between V_{m+1}, f_{m+1} and V_{n*}, f_{n*}, we have $z(f_{m+1}(v)) = f_{n*}(v)$. Putting these two facts together gives $z(i_{n,m}(f_n(v))) = f_{n*}(v)$ as desired.

$t_m((\forall x)\phi(x))$ as desired. $\square$

13.2 Vindictation of FOL

Now let us return to our ultimate task: showing that every first-order logical argument in the language of set theory can be transformed into an argument in the deduction system described above, which takes us from the translation of the premises for this argument to the translation of its conclusion.

Proposition 13.2.1. *If $\gamma_1, \ldots, \gamma_m, \theta$ are sentences in the language of set theory and $\gamma_1 \ldots \gamma_m \vdash_{FOL} \theta$ then $t(\gamma_1) \ldots t(\gamma_m) \vdash (\theta)$ (where $\vdash_{FOL}$ represents provability in first order logic and $\vdash$ represents provability in the formal system just introduced).*

Proof. First note that it suffices to show that whenever $\gamma_1, \ldots, \gamma_m, \theta$ are as in the proposition above we also have $\mathscr{V}(V_0), t_0(\gamma_1) \ldots t_0(\gamma_m) \vdash t_0(\theta)$. For, by the lemma below, any such proof can be transformed into a proof one witnessing $t(\gamma_1) \ldots t(\gamma_m) \vdash t(\theta)$.

Lemma 13.2.2. *It $\mathscr{V}(V_0), t_0(\gamma_1) \ldots t_0(\gamma_m) \vdash t_0(\theta)$ then $t(\gamma_1) \ldots t(\gamma_m) \vdash t(\theta)$.*

Proof. Recall that when ψ is a sentence of set theory, $t(\psi) = \Box(\mathscr{V}(V_0) \rightarrow t_0(\psi))$. If we have a proof witnessing $\mathscr{V}(V_0), t_0(\gamma_1) \ldots t_0(\gamma_m) \vdash t_0(\theta)$, then we also have one witnessing $\vdash \mathscr{V}(V_0) \wedge t_0(\gamma_1), \ldots, t_0(\gamma_m) \rightarrow t_0(\theta)$. The latter proof can be transformed into one witnessing $t(\gamma_1) \ldots t(\gamma_m) \vdash t(\theta)$ as follows.

Start with the assumptions that $t(\gamma_1) \ldots t(\gamma_m)$, i.e., $\Box(\mathscr{V}(V_0) \rightarrow t_0(\gamma_i))$ for each i. To get the desired conclusion that $t(\theta) = \Box(\mathscr{V}(V_0) \rightarrow t_0(\theta))$, we will make a $\Box$I argument. Consider an arbitrary logically possible scenario (holding fixed nothing). Suppose, for conditional introduction, that $\mathscr{V}(V_0)$. Our assumptions $t(\gamma_i)$ are content-restricted to the empty set (because their outer connective is a $\Box$ with no subscripts), so they can be assumed to remain true within the arbitrary logically possible scenario under consideration. By applying $\Box$E to each $t(\gamma_i)$, we get the corresponding $\mathscr{V}(V_0) \rightarrow t_0(\gamma_i)$ claim. Thus we can derive that $t_0(\gamma_i)$.

By the assumptions in the first paragraph we can also prove from empty premises that $\mathscr{V}(V_0) \wedge t_0(\gamma_1), \ldots, t_0(\gamma_m) \rightarrow t_0(\theta)$. Thus we can infer that $t_0(\theta)$. Discharging our assumption that $\mathscr{V}(V_0)$ gives us the conditional $\mathscr{V}(V_0) \rightarrow t_0(\theta)$ within the logically possible scenario being considered.

Since all of our assumptions are content-restricted to the empty set, the rules for $\Box$I allow us to conclude that $t(\theta) = \Box[\mathscr{V}(V_0) \rightarrow t_0(\theta)]$, as desired. $\Box$

Thus it suffices to show that if $\mathscr{V}(V_0), t_0(\gamma_1) \ldots t_0(\gamma_m) \vdash t_0(\theta)$ then $\mathscr{V}(V_0), t_0(\gamma_1) \ldots t_0(\gamma_m) \vdash t_0(\theta)$.

We will prove this claim by induction on the length of FOL proofs. But first I must explain what I mean by FOL proofs.

Given a rule such as ($\wedge$I) If $\Gamma_1 \vdash_{FOL} \theta$ and $\Gamma_2 \vdash_{FOL} \psi$, then $\Gamma_1, \Gamma_2 \vdash_{FOL} (\theta \wedge \psi)$ we refer to $\Gamma_1 \vdash_{FOL} \theta$ and $\Gamma_2 \vdash_{FOL} \psi$ as the first and second inputs to the rule, $\Gamma_1, \Gamma_2 \vdash_{FOL} (\theta \wedge \psi)$ as the output of the rule.

Definition 13.2.3. I will take FOL proofs to be finite sequences of numbered lines, where each line corresponds to an application of one of our FOL closure conditions. Specifically, a proof is a sequence of lines where the n-th line is given by $\langle \theta_n, R_n, \vec{l}_n, \Gamma_n \rangle$, such that θ_n is a formula of FOL, Γ_n is a set of first order formulas and $\vec{l}$ is a tuple of line numbers $c_i \leq n$ where i indexes the inputs required by rule $\boldsymbol{R}$ (where we regard the assumption rule as having itself as an input). So, for example, if $\boldsymbol{R}$ was $\wedge \boldsymbol{I}$ the tuple $\vec{l}$ would be a pair of line numbers while if $\boldsymbol{R}$ was $\boldsymbol{As}$ it would be empty. Finally, we require that $\boldsymbol{R}$ yield $\Gamma_n \vdash \theta_n$ when acting on the inputs $\Gamma_{c_i} \vdash \theta_{c_i}$ for c_i in $\vec{l}$. Note that the validity of a proof is a simple syntactic matter to check. I will say that a proof **witnesses** the fact that $\Gamma \vdash_{FOL} \theta$ if its last line takes the form $\langle \theta, \boldsymbol{R}(...), \vec{l}_n, \Gamma \rangle$.

Note that it follows from the definition above that truncating an n-line proof of θ at some line $m < n$ $\langle \theta_{n'}, \boldsymbol{R}(c_1, c_2), \Gamma_{n'} \rangle$ yields a strictly shorter m-line proof that $\Gamma_{n'} \vdash_{FOL} \theta_{n'}$.

I will proceed by induction on the length of the proof to be translated.

We first establish the claim holds for proofs of height 1. Only two kinds of proofs of length 1 are possible.

First, there are proofs generated by a single application of the assumption introduction rule, i.e., proofs witnessing $\phi \vdash_{FOL} \phi$. Clearly in all such cases we have $\mathscr{V}(V_0), t_0(\phi) \vdash t_0(\phi)$.

Second, there single line proofs of $\vdash_{FOL} v = v$ produced by an application of (=I). In all such cases we have $\mathscr{V}(V_0), \vdash t_0(v = v)$, since $t_0(v = v)$ is $f_0(v) = f_0(v)$.

Now assume, by way of induction, that whenever θ is any *formula* in the language of set theory, Γ is a set of such formulas and there is a proof of θ from Γ with height at most n, then $\mathscr{V}(V_0), \Gamma \vdash t_0(\theta)$ and establish the claim for proofs of height $n + 1$

We consider each possibility for the final rule used in the proof of θ

We first note that since our definition of t_n above (specifically, given the fact that we have defined t_0 to commute with $\wedge, \vee, \neg, \rightarrow$), it is trivial to demonstrate the inductive step in every case where the proof ends with one of the rules below.

($\wedge$I) If $\Gamma_1 \vdash_{FOL} \theta$ and $\Gamma_2 \vdash_{FOL} \psi$, then $\Gamma_1, \Gamma_2 \vdash_{FOL} (\theta \wedge \psi)$.

($\wedge$E) If $\Gamma \vdash_{FOL} (\theta \wedge \psi)$ then $\Gamma \vdash_{FOL} \theta$; and if $\Gamma \vdash_{FOL} (\theta \wedge \psi)$ then $\Gamma \vdash_{FOL} \psi$.

($\vee$I) If $\Gamma \vdash_{FOL} \theta$ then $\Gamma_1 \vdash_{FOL} h\, \theta \vee \psi$; if $\Gamma \vdash_{FOL} \psi$ then $\Gamma \vdash_{FOL} \theta \vee \psi$.

($\vee$E) If $\Gamma_1 \vdash_{FOL} (\theta \vee \psi)$, $\Gamma_2, \theta \vdash_{FOL} \phi$ and $\Gamma_3, \psi \vdash_{FOL} \phi$, then $\Gamma_1, \Gamma_2, \Gamma_3 \vdash_{FOL} \phi$.

($\rightarrow$ I) If $\Gamma, \theta \vdash_{FOL} \psi$, then $\Gamma \vdash_{FOL} (\theta \rightarrow \psi)$.

($\rightarrow$ E) If $\Gamma_1 \vdash_{FOL} (\theta \rightarrow \psi)$ and $\Gamma_2 \vdash \theta$, then $\Gamma_1, \Gamma_2 \vdash \psi$.

($\neg$ I) If $\Gamma_1, \theta \vdash_{FOL} \psi$ and $\Gamma_2, \theta \vdash_{FOL} \neg\psi$, then $\Gamma_1, \Gamma_2 \vdash \neg\theta$.

(DNE) If $\Gamma \vdash_{FOL} \neg\neg\theta$ then $\Gamma \vdash_{FOL} \theta$.

($\perp$ I) If $\Gamma \vdash_{FOL} \psi \wedge \neg\psi$ then $\Gamma \vdash \perp$.

($\perp$ E) If $\Gamma, \theta_{FOL} \vdash \perp$ then $\Gamma \vdash_{FOL} \neg\theta$.

For example, to check the clause for $\rightarrow$I, suppose that we have a length n+1 proof of $\theta \rightarrow \psi$ whose final rule is $\rightarrow$I. This ensures that we have a height-n proof of ψ from the hypotheses Γ, θ. By inductive hypothesis, we have a proof of $t_0(\psi)$ from $\mathscr{V}(V_0), t_0[\Gamma], t_0(\theta) \vdash t_0(\psi)$. Adding an application of $\rightarrow$I to the end of this proof produces a proof of $\mathscr{V}(V_0), t_0[\Gamma] \vdash t_0(\theta) \rightarrow t_0(\psi)$.

Now, consider the possibility that our proof of length $n+1$ concludes with an application of $\forall$I.

($\forall$I) If $\Gamma \vdash_{FOL} \theta$ and the variable v does not occur free in any member of Γ, then $\Gamma \vdash_{FOL} \forall v\theta$.

Given an $n+1$-line proof of $\forall v\theta$, we can extract an n-line proof witnessing $\gamma_1 ... \gamma_n \vdash_{FOL} \theta$ where the variable v does not occur free in any member of Γ. So, by inductive hypothesis, we have $\mathscr{V}(V_0), t_0[\Gamma] \vdash t_0(\theta)$ and we must establish $\mathscr{V}(V_0), t_0[\Gamma] \vdash t_0(\forall v\theta) = \Box_{V_0}[V_1 \geq_{\mathbf{v}} V_0 \rightarrow t_1(\theta)]$.

So assume $\mathscr{V}(V_0), t_0[\Gamma]$. Now consider an arbitrary scenario in which $V_1 \geq_{\mathbf{v}} V_0$ (while the facts about V_0 are held fixed). If we could infer $t_1(\theta)$ the result would follow immediately.

But the inductive hypothesis (and the fact that $\mathscr{V}(V_0)$ is content restricted to V_0, and by observation 2 in chapter 6.2, so is $t_0[\Gamma]$) we can infer that $t_0(\theta)$. Since v is free in θ in different choices for $V_1 \geq_{\mathbf{v}} V_0$ could (in principle) change the truth value of $t_1(\theta)$.

However, one important further fact is available: our inductive hypothesis says that $t_0(\theta)$ is derivable from $\mathscr{V}(V_0), t_0[\Gamma]$ and these formula make no mention of V_0's assignment of v. Because the elements of Γ don't mention v as a free variable (and this is the only variable on which f_1 is allowed to differ from f_2 $t_0(\gamma)$ cannot change when we go from considering V_0 to V_1: by the translation lemma we have $V_1 \geq V_0 \rightarrow (t_1[\gamma_i] \leftrightarrow t_0[\gamma_i])$, hence $t_1[\Gamma]$. Observation 3 in chapter 6.2, says that derivability facts don't change when we swap subscripts, so we have $\mathscr{V}(V_1), t_1[\Gamma] \vdash t_1(\theta)$ because we have $\mathscr{V}(V_0), t_0[\Gamma] \vdash t_0(\theta)$. Thus we can deduce that $t_1(\theta)$, as desired.

Thus, $\mathscr{V}(V_0), t_0[\Gamma] \vdash V_1 \geq_{\mathbf{v}} V_0 \rightarrow t_1(\theta)$. However, as $\mathscr{V}(V_0), t_0[\Gamma]$ are content restricted to V_0 by $\Box$I we derive $\mathscr{V}(V_0), t_0[\Gamma] \vdash \Box_{V_0}[V_1 \geq_{\mathbf{v}} V_0 \rightarrow t_1(\theta)]$.

The last two clauses we need to check can be handled very quickly, once we prove the following lemma.

Lemma 13.2.4. *For any formula ϕ, $\vdash \mathscr{V}(V_i) \wedge f_i(v) = f_i(v') \rightarrow (t_i(\phi) \leftrightarrow t_i(\phi'))$, where ϕ' is obtained from ϕ by replacing zero or more occurrences of v (not necessarily all such uses) with v', provided that no bound variables are replaced, and all substituted occurrences of v' are free.*

Proof. I will argue by induction on the complexity of the formula ϕ.

This fact is immediate in the base case where ϕ has the form $v = v'$ or $v \in v'$, (for some variables v and v', not necessarily distinct).

It is also immediate that if this claim holds for ρ, ψ then it holds for $\phi = \neg\rho$, $\phi = \rho \wedge \psi$ etc.

Suppose $\phi = (\exists x)(\psi)$ and the result above is true for ψ. There are two possibilities to consider.

First consider the case where x is neither v_1 or v_2. In this case we have $t_i(\phi') = \Diamond_{V_i}[V_{i+1} \geq_x V_i \wedge t_{i+1}(\psi')]$. To establish the bi-conditional it is enough to show we can derive $t_i(\phi')$ from $t_i(\phi)$ and vice versa. We do this by entering the diamond context, importing $f_i(v_1) = f_i(v_2)$ and using the inductive hypothesis to infer $t_{i+1}(\psi)$ from $t_{i+1}(\psi')$ (and vice versa).

If $x = v_1$ then $\phi[v_1/v_2] = (\exists v_1 \psi)[v_1/v_2] = \phi$.[4] If $x = v_2$, by hypothesis v_1 may not appear free in ϕ least v_2 be bound in $\phi[v_1/v_2]$, so again $\phi[v_1/v_2] = \phi$.

Exactly analogous reasoning works when $\phi = (\forall x)(\psi)$.

□

[4]Remember that $[v_1/v2]$ only substitutes v_1 for v_2 in cases where v_1 occurs free.

Next, suppose that our proof tree of height $n + 1$ concludes with an application of ∀E.

(∀E) If $\Gamma \vdash \forall v\theta$, then $\Gamma \vdash \theta(v|v')$, provided that v is free for v' in θ.

Then there is proof with height at most n of $\Gamma \vdash \forall v\theta$. From this we must establish $\mathscr{V}(V_0), t_0(\Gamma) \vdash t_0(\theta(v|v'))$.

So assume $\mathscr{V}(V_0), t_0(\Gamma)$. By the inductive hypothesis we can infer $\Box_{V_0}[V_1 \geq_{\mathbf{v}} V_0 \to t_1(\theta)]$. We note that, holding V_0 fixed it is logically possible that $V_1 \geq_{\mathbf{v}} V_0$ and $f_1(v)$ to $f_0(v')$. We can import $\Box_{V_0}[V_1 \geq_{\mathbf{v}} V_0 \to t_1(\theta)]$ into this logical possibility context and thereby infer $\Diamond_{V_0} t_1(\theta)$.

Because $f_1(v) = f_0(v') = f_1(v')$, we can derive that $t_1(\theta(v|v'))$ must also be true in this scenario by the lemma above. The fact that v is free for v' in θ means that all the substituted instances of v' are free (so the requirements for this Lemma 13.2.4 are fulfilled). Now we can use the Translation Lemma to derive that $t_0(\theta(v|v'))$ is also true in this scenario, as follows. Recall that Translation Lemma tells us that when $v_1 ... v_k$ are all the free variables in θ we have $\vdash V_n \geq_{set} V_m \wedge f_n = f_m(v_1) \wedge ... f_n(v_k) = f_m(v_k) \to (t_n(\phi) \leftrightarrow t_m(\phi))$.

By definition, $\theta(v|v')$ substitutes v' for every free instance of v in θ. So v (the only variable on which f_1 and f_0 can disagree) never occurs free in $\theta(v|v')$. So we can derive any sentence of this form in any context, including inside our current $\Diamond$ context. Thus we can derive that $t_0(\theta(v|v')) \leftrightarrow t_1(\theta(v|v'))$, and hence infer that $t_0(\theta(v|v'))$.

Thus we have $\Diamond_{V_0} t_0(\theta(v|v'))$. Finally we can use the fact that $t_0(\theta(v|v'))$ is content-restricted to V_0, f_0 to infer the truth of $t_0(\theta(v|v'))$ simpliciter, as desired.

Finally, consider the possibility that our proof of height $n+1$ concludes with an application of =E.

(=E) If $\Gamma_1 \vdash v_1 = v_2$ and $\Gamma_2 \vdash \theta$, then $\Gamma_1, \Gamma_2 \vdash \theta'$, where θ' is obtained from θ by replacing zero or more occurrences of v with v' [(so unlike in $\theta(v|v')$ not all free instances of v must be replaced)], provided that no bound variables are replaced, and all substituted occurrences of v' are free.

In this case, we have a proof with height 1 for $\Gamma_1 \vdash_{FOL} v_1 = v_2$ and one of height n for $\Gamma_1, \vdash_{FOL} \theta'$. So by inductive hypothesis, we have $\mathscr{V}(V_0), t_0(\Gamma_1) \vdash t_0(v = v')$, i.e., $f_0(v) = f_0(v')$ and $\mathscr{V}(V_0), t_0(\Gamma_1)) \vdash t_0(\theta)$.

So assume $\mathscr{V}(V_0), t_0(\Gamma_1), t_0(\Gamma_2)$. Then we can derive $t_0(\theta)$ and $f_0(v_1) = f_0(v_2)$, by the fact just noted. We need to show that $t_0(\theta')$ can be derived. As required by the Lemma above, we know that θ' is obtained from θ by replacing zero or more occurrences of v_1 with v_2, where all substituted occurrences of v_2 are free. Thus we can prove $\mathscr{V}(V_0) \to (t_0(\theta) \leftrightarrow t_0(\theta'))$ from empty premises. So we can can conclude that $t_0(\theta')$.

Thus we have $\mathscr{V}(V_0), t_0(\Gamma_1)), t_0(\Gamma_2)) \vdash t_0(\theta')$, as desired.

□

Chapter 14

Defense of the ZFC Axioms

Finally, it remains to show that my potentialist translations of the ZFC axioms of set theory can be proved using my inference rules for logical possibility.

I will frequently use iterated applications of the $\Box$ and $\Diamond$ Collapsing Lemmas (proved in sections 11.1 and 11.3) to simplify the translation of set theoretic sentences. Recall that the $\Box$ Collapsing Lemma says:

"If ϕ_2 and θ are content restricted to $\mathcal{L}_1, \mathcal{L}_2$ and ϕ_1 is content restricted to $\mathcal{L}_0, \mathcal{L}_1$, then we have

$\vdash \Box_{\mathcal{L}_0}(\phi_1 \rightarrow \Box_{\mathcal{L}_1}(\phi_2 \rightarrow \theta)) \leftrightarrow \Box_{\mathcal{L}_0}(\phi_1 \land \phi_2 \rightarrow \theta)$"

This lets us simplify the translation of set theoretic statements with repeated $\forall$ quantifiers by replacing a string of $\Box$ statements with a single $\Box$ statement (and similarly with $\Diamond$ statements.[1].

So, for instance, a set theoretic claim of the form $(\forall x)(\forall y)(\phi)$ gets trans-

[1]Translations for strings of repeated $\exists$ quantifiers which becomes strings of $\Diamond$ statements are collapsed into a single $\Diamond$ using the $\Diamond$ collapsing lemma similarly

lated as follows,

$\Box(\mathscr{V}(\vec{V}_0) \to \Box_{\vec{V}_0}[\vec{V}_1 \geq_{\mathbf{x}} \vec{V}_0 \to \Box_{\vec{V}_1}(\vec{V}_2 \geq_{\mathbf{y}} \vec{V}_1 \to t_2(\phi))])$

However, it is provably equivalent to the following simpler sentence, via two applications of the $\Box$ Simplification Lemma[2]. (The fact that that the sentence inside each $\Box_{V_i}$ or $\Diamond_{V_i}$ subformula in the translation of a set theoretic sentence ϕ is always content-restricted to V_i, V_{i+1} ensures that the premises of the above Lemma are satisfied).

$\Box(\mathscr{V}(\vec{V}_0) \wedge \vec{V}_1 \geq_{\mathbf{x}} \vec{V}_0 \wedge \vec{V}_2 \geq_{\mathbf{y}} \vec{V}_1 \to t_2(\phi)])$

In what follows, I will sketch the reasoning used to prove relevant propositions, but leave it to the reader to fill in the technical details such as applying the wrapping trick or subscripting relations to mimic quantifying in.

[note that by my abbreviations in $f(y) = y$, ONLY the right hand token is a genuine variable]

14.1 Foundation and Other Easy Cases

Proposition 14.1.1. ***Foundation*** $(\forall x)[(\exists a)(a \in x) \to (\exists y)(y \in x \wedge \neg(\exists z)(z \in y \wedge z \in x))]$ *Translating this and then simplifying with $\Diamond$-Collapsing Lemma as above yields:* $\Box[\mathscr{V}(\vec{V}_0) \wedge \vec{V}_1 \geq_{\boldsymbol{x}} \vec{V}_0 \wedge \Diamond_{\vec{V}_1}[\vec{V}_2 \geq_{\boldsymbol{a}} \vec{V}_1 \wedge f_2(a) \in f_2(x)] \to \Diamond_{\vec{V}_1}[\vec{V}_2 \geq_{\boldsymbol{y}} \vec{V}_1 \wedge f_2(y) \in f_2(x) \wedge \neg\Diamond_{\vec{V}_2}(\vec{V}_3 \geq_{\boldsymbol{z}} \vec{V}_2 \wedge f(z) \in_3 f_3(y) \wedge f_3(z) \in_3 f_3(x))])]$

This essentially says: if V_1, f_1 can be extended such that $f_1(a)$ is $\in_2$ $f_2(x)$, then it could alternatively be extended by a V_2, f_2 whose assignment for

[2] The trick is to first use the $\Box$ Simplifying Lemma to simplify the innermost statement, in this case, $\Box_{V_0}[V_1 \geq_{\mathbf{x}} V_0 \to \Box_{V_1}(V_2 \geq_{\mathbf{y}} V_1 \to \phi)]$, and then to proceed progressively outward [SAY MORE?]

y ensures that no further extension V_3, f_3 can assign f_3 of z such that $f_3(z) \in_3 f_3(y) \wedge f_3(z) \in_3 f_3(x)$.

To this end, we prove the following lemma.

Lemma 14.1.2. $\mathscr{V}(V) \rightarrow (\forall x)[(\exists a)(a \in x) \rightarrow (\exists y)(y \in x \wedge \neg(\exists z)(z \in y \wedge z \in x))]$

Proof. Assume that $\mathscr{V}(V)$. Consider an arbitrary x, such that $set(x)$ and $(\exists a)(a \in x)$. By the fact that the ords are well ordered by $\leq$ (as defined in 9.1.1), there will be some $\leq$-least member of ord o with the following property: there exists y at level o and $y \in x$. Any $z \in y$ occurs at some level $o' < o$, by the fact that $\mathscr{V}(V)$. Thus, by minimality of o, $\neg z \in x$. Thus we have $y \in x$ such that $\neg(\exists z)(z \in y \wedge z \in x)$, as desired. □

Proof. Now we will prove the proposition using the lemma above. Consider an arbitrary situation in which $\mathscr{V}(\vec{V}_0) \wedge \vec{V}_1 \geq_{\mathbf{x}} \vec{V}_0 \wedge \Diamond_{\vec{V}_1}[\vec{V}_2 \geq_{\mathbf{a}} \vec{V}_1 \wedge f_2(a) \in f_2(x)]$.

Note that if $f_1(x)$ is the empty set, then it is not possible (fixing the facts about $\vec{V}_1$) to have $\vec{V}_2 \geq_a \vec{V}_1$ with $f_2(a) \in f_2(x)$. Thus, we may assume $f_1(x)$ is not the empty set. Thus, by the above lemma (and simplified choice), we can choose a y such that $y \in_1 x \wedge \neg(\exists z)(z \in_1 y \wedge z \in_1 x)$.

[can finish by just using new lemma here]

We can then let $V_2 = V_1$ and f_2 to be just like f_1, except that $f_2(y) = y$. Thus we have $f_2(y) \in f_2(x) \wedge (\forall z)\neg(z \in_2 f_2(y) \wedge z \in_2 f_2(x)$.

Now, suppose for contradiction that it were $\Diamond_{\vec{V}_2}$ to have $\vec{V}_3 \geq_z \vec{V}_2$ with $f_3(z) \in_3 f_3(y) \wedge f_3(z) \in_3 f_3(x)$. Then we would have $f_3(z) \in_3 f_2(x)$ and

$f_3(z) \in_2 f_2(y)$ [by the fact that $\vec{V_3} \geq_z \vec{V_2}$]. But this contradicts our choice for $f_2(y)$, specifically, the fact that $(\forall z)\neg(z \in_2 f_2(y) \wedge z \in_2 f_2(x)$[3].

Thus we can conclude that $\Diamond_{\vec{V_1}}[\vec{V_2} \geq_{\mathbf{y}} \vec{V_1} \wedge f_2(y) \in f_2(x) \wedge \neg\Diamond_{\vec{V_2}}(\vec{V_3} \geq_{\mathbf{z}} \vec{V_2} \wedge f(z) \in_3 f_3(y) \wedge f_3(z) \in_3 f_3(x))]$, as desired.

□

Potentialist versions of Extensionality, Pairing, Powerset, Union and Choice can be proved in much the same way noted above, by using the fact that the corresponding principle must hold within any initial segment V_i such that $\mathscr{V}(V_i)$.

Proposition 14.1.3 (Extensionality). $(\forall x)(\forall y)[(\forall z)(z \in x \leftrightarrow z \in y) \to x = y]$ *Translating this and then simplifying via* $\Box$ *collapsing (and a little FOL) yields* $\Box[\mathscr{V}(\vec{V_0}) \wedge \vec{V_1} \geq_x \vec{V_0} \wedge \vec{V_2} \geq_y \vec{V_1} \wedge \Box_{\vec{V_2}}(\vec{V_3} \geq_z \vec{V_2} \to [f_3(z) \in_3 f_3(x) \leftrightarrow f_3(z) \in_3 f_3(y)]) \to f_2(x) = f_2(y)]$

Informally, this says that $f_2(x)$ *and* $f_2(y)$ *are assigned in such a way that any extending* $\vec{V_3} \geq_z \vec{V_2}$ *must satisfy* $f_3(z) \in f_3(x) \leftrightarrow f_3(z) \in f_3(y)$*, then* $f_2(x) = f_2(y)$.

Proof. I will prove this claim by exploiting the fact that extensionality holds *inside* any relevant V_2 such that $\mathscr{V}(\vec{V_2})$ (because Thinness includes an extensionality requirement) to argue that $f_2(x) = f_2(y)$.

Assume that $\vec{V_0}, \vec{V_1}, \vec{V_2}$ satisfy $\mathscr{V}(\vec{V_0}) \wedge \vec{V_1} \geq_{\mathbf{x}} \vec{V_0} \wedge \vec{V_2} \geq_{\mathbf{y}} \vec{V_1}$ and $\Box_{\vec{V_2}}(\vec{V_3} \geq_{\mathbf{z}} \vec{V_2} \to [f_3(z) \in_3 f_3(x) \leftrightarrow f_3(z) \in_3 f_3(y)])$.

Now suppose for contradiction that$\neg f_2(x) = f_2(y)$. By the fact that V_2 satisfies extensionality there is some $set_2(k)$ such that $\neg(k \in_2 f_2(x) \leftrightarrow k \in_2$

[3]Note that this sentence is content restricted to V_2 so it must remain true in our current context

$f_2(y))$. Thus, it is possible (holding $\vec{V}_2$ fixed) that $\vec{V}_3 \geq_{\mathbf{z}} \vec{V}_2$ and $f_3(\mathbf{z})$ applies to such a set_2 k.[4] However, (by unpacking the definition of $\vec{V}_3 \geq_z \vec{V}_2$) it follows that this scenario must be one in which $\neg[f_3(z) \in_3 f_3(x) \leftrightarrow f_3(z) \in_3 f_3(y)]$), contrary to the $\Box_{\vec{V}_2}$ assumption above.[5]

Thus, we have a our desired proof by contradiction that $f_2(x) = f_2(y)$. And since $\vec{V}_0, \vec{V}_1, \vec{V}_2$ are arbitrary, we can derive that the above statement holds with logical necessity.[6] $\Box$

Proposition 14.1.4 (Union). *"$\forall z\, \exists a\, \forall y\, \forall x[(x \in y \wedge y \in z) \Rightarrow x \in a]$."*

Translating and then applying the $\Box$ Collapsing Lemma gives

$\Box(\mathscr{V}(\vec{V}_0) \wedge \vec{V}_1 \geq_z \vec{V}_0 \rightarrow \Diamond_{\vec{V}_1}[\vec{V}_2 \geq_{\boldsymbol{a}} \vec{V}_1 \wedge \Box_{\vec{V}_2}(\vec{V}_3 \geq_{\boldsymbol{x}} \wedge \vec{V}_2 \wedge V_4 \geq_{\boldsymbol{y}} \vec{V}_3 \rightarrow [f_4(x) \in_4 f_4(y) \wedge f_4(y) \in f_4(z) \rightarrow f_4(x) \in f_4(a)])])$.

Thus it essentially says that for any V_1, f_1 assigning z, there is an extension V_2, f_2 which assigns $\boldsymbol{a}$ to a 'union set' for $f_1(z)$.[7]

Proof. As before, we will prove the needed conclusion by exploiting the fact that Union holds true within *within* any V_1 such that $\mathscr{V}(\vec{V}_1)$. Consider

[4] We know this by Simple Choice and the Multiple Definitions Lemma.

[5] Specifically, by the Simpler Choice Lemma it is logically possible that the otherwise unused predicate $\boldsymbol{P}(z)$ applies to a unique object z satisfying the formula above. Entering this $\Diamond_{\vec{V}_2,\boldsymbol{P}}$ context and applying simple comprehension a few times (as per the Multiple Definitions Lemma), it is logically possible that $V_3 =_{set} V_2$ and $(\forall k)(f_3(\mathbf{z}) = k \leftrightarrow \boldsymbol{P}(z))$ and that $(\forall y)(\neg y = \mathbf{z}) \rightarrow f_3(y) = f_2(y))$ for all other values of y.

Enter this $\Diamond_{\vec{V}_0,\vec{V}_1,\vec{V}_2,\boldsymbol{P}}$ context. The fact that $(\forall k)[\boldsymbol{P}(k) \rightarrow \neg(z \in_2 f_2(x) \leftrightarrow z \in_2 f_2(y))]$ is content-restricted to V_2 so it can be imported into this context. Combining this with our specification that $f_3(\mathbf{z})$ is the unique object satisfying $\boldsymbol{P}(x)$ and $f_3 = f_2$ on all other values, we get $\neg[\vec{V}_3 \geq_{\mathbf{z}} \vec{V}_2 \rightarrow [f_3(\mathbf{z}) \in_3 f_3(\mathbf{x}) \leftrightarrow f_3(\mathbf{z}) \in_3 f_3(\mathbf{y})]$.

Leaving this $\Diamond_{\vec{V}_0,\vec{V}_1,\vec{V}_2,\boldsymbol{P}}$ context, Inn$\Diamond$ allows us to conclude that $\Diamond_{\vec{V}_1,\vec{V}_2}\neg[\vec{V}_3 \geq_{\mathbf{z}} \vec{V}_2 \rightarrow [f_3(z) \in_3 f_3(x) \leftrightarrow f_3(z) \in_3 f_3(y)]$. But our $\Box_{\vec{V}_1,\vec{V}_2}(\vec{V}_3 \geq_{\mathbf{z}} \vec{V}_2 \ldots)$ assumption above is the negation of this claim.

[6] That is, we can derive this conclusion via $\Box$I because all the 0 assumptions used to secure this result are content restricted to the empty list.

[7] in the sense that extending V_3, f_3, V_4, f_4 which assign x and then y so that $f_4(x) \in_4 f_4(y) \in_4 f_4(z)$ must also satisfy $f_4(x) \in_4 f_4(a)$)

an arbitrary scenario in which $\mathscr{V}(\vec{V_0}) \wedge \vec{V_1} \geq_{\mathbf{z}} \vec{V_0}$. We can derive the fact that there is unique $set_1(w)$ such that $(\forall k)[k \in_1 w \leftrightarrow (\exists k')(k \in_1 k' \wedge k' \in f_1(x)]$ from the fact that $\mathscr{V}(V_1)$ as follows. It is logically possible that, [review wording] letting $\boldsymbol{H}$ stand fo some otherwise-unused one place relation symbol, (given the facts about $\vec{V_1}$) that $(\forall k)(\boldsymbol{H}(k) \leftrightarrow \exists k' k \in_1 k' \wedge k' \in f_1(x))$ by comprehension. We can deduce that $f_1(o)$ occurs at some ordinal level and everything everything that satisfies $\boldsymbol{H}$ occurs at a lower level than $\boldsymbol{o}$. Thus, by the thickness property of $\mathscr{V}(\vec{V_1})$, we have that there is a set_1 w occuring at level $\boldsymbol{o}$ which contains exactly the elements of H. Thus we have that there is a $set_1(w)$ such that $(\forall k)(k \in_1 w \leftrightarrow (\exists k')k \in_1 k' \wedge k' \in f_1(x))$. Now the above claim is this sentence is content-restricted to V_1 it must have been true in our original scenario.

Thus, there is a$set_1(w)$ which behaves like a union set for $f_1(x)$ as above. By Simple Comprehension (and the Multiple Definition Lemma) and it is logically possible (given the facts about V_1, f_1) to have $\vec{V_2} \geq_a \vec{V_1}$ such that $V_2 =_{set} V_1$ and $f_2(a)$ is this set.[8]

It now is straightforward to verify that V_1, V_2 witness the desired relationship.[9] □

Proposition 14.1.5 (Pairing)**.** *"$\forall x \forall y \exists z(x \in z \wedge y \in z)$" Translating and then applying the □ Collapsing Lemma gives* $\Box[\mathscr{V}(\vec{V_0}) \wedge \vec{V_1} \geq_{\boldsymbol{x}} \vec{V_0} \wedge \vec{V_2} \geq_{\boldsymbol{y}}$

[8]i.e, it is the unique set_1 w such that $(\forall k)(k \in_1 w \leftrightarrow (\exists k')k \in_1 k' \wedge k' \in f_1(x))$

[9] To check that this choice for $f_2(a)$ behaves as desired, consider an arbitrary scenario (holding facts about $\vec{V_2}$ fixed) in which $\vec{V_3} \geq_{\mathbf{x}} \vec{V_2} \wedge V_4 \geq_{\mathbf{y}} \vec{V_3}$ such that $f_4(x) \in_4 f_4(y) \wedge f_4(y) \in_4 f_4(z))$. By the fact that $f_4(z) = f_3(z) = f_2(z)$ we have $f_4(y) \in_2 f_2(z)$, and thus $f_4(x) \in_2 f_4(y)$. Now, our characterization of $f_2(a)$ above is content-restricted to $\vec{V_2}$, so it must remain true in the current context. Thus we have: $(\forall k)(k \in_2 f_2(a) \leftrightarrow (\exists k')k \in_2 k' \wedge k' \in f_2(z))$. Putting these facts together we can derive $f_4(x) \in_2 f_2(a)$ and hence $f_4(x) \in_4 f_4(a)$, as desired.

The reader can now see how the result follows.

$\vec{V}_1 \to \Diamond_{\vec{V}_1}(\vec{V}_3 \geq_z \vec{V}_3 \wedge f_3(x) \in_3 f_3(z) \wedge f_3(y) \in_3 f_3(z)))$

Thus it essentially says that any V_2, f_2 assigning x and y can be extended by a V_3, f_3 assigning z such that $f_3(z)$ contains exactly $f_2(x)$ and $f_2(y)$

Proof. Consider an arbitrary situation in which $\mathscr{V}(\vec{V}_0) \wedge \vec{V}_1 \geq_{\mathbf{x}} \vec{V}_0 \wedge \vec{V}_2 \geq_{\mathbf{y}} \vec{V}_1$.

By the fact that $\mathscr{V}(V_2)$ and the One More Layer Lemma**??**, we can have (while holding fixed the facts about V_2) a V_3 which extends V_2 by adding one more layer of classes [10]. By simple comprehension, it would be possible (holding fixed the facts about $\vec{V}_2, \vec{V}_3$) for a predicate $\boldsymbol{P}$ to apply to exactly those objects z such that $z = f_2(x) \vee z = f_2(y)$. Thus V_3 includes a (unique) set_3 k whose sole elements are $f_2(x)$ and $f_2(y)$.[11] Now by Simple Comprehension and the Multiple Definitions Lemma, it is $\Diamond_{\vec{V}_2}$ to have f_3 such that $\vec{V}_3 \geq_{\mathbf{z}} \vec{V}_3$ except for $f_3(z) =$ the unique set_3 whose elements are exactly $f_2(x)$ and $f_2(y)$.

Entering this $\Diamond_{\vec{V}_2}$ context and using first order logic to unpack definitions yields the desired conclusion that $\vec{V}_3 \geq_{\mathbf{z}} \vec{V}_3 \wedge f_3(x) \in_3 f_3(z) \wedge f_3(y) \in_3 f_3(z)$.

Exiting this $\Diamond_{\vec{V}_2}$ context [inc? and pulling out the above, suitably content-restricted conclusion], and completing our conditional argument yields $\mathscr{V}(\vec{V}_0) \wedge \vec{V}_1 \geq_{\mathbf{x}} \vec{V}_0 \wedge \vec{V}_2 \geq_{\mathbf{y}} \vec{V}_1 \to \Diamond_{\vec{V}_2}(\vec{V}_3 \geq_{\mathbf{z}} \vec{V}_3 \wedge f_3(x) \in_3 f_3(z) \wedge f_3(y) \in_3 f_3(z))$. Finally, since we proved this from empty assumption, it holds with logical necessity, as above. □

[10] In essence this is a scenario where we have a layer of classes over all the objects in $Ext(V_2)$ and then take set_3 apply to all the set_2s plus all of the classes which are not co-extensive with some already existing set_2, and define everything else in the obvious way

[11] [make full sntence]since there is a class with this property, and that class is either a set_3 itself or has exactly the same elements as some set_2

Proposition 14.1.6 (Powerset)**.** *"$\forall x \exists y \forall z[z \subseteq x \to z \in y]$" That is, "$\forall x \exists y \forall z[(\forall w)(w \in z \to w \in x) \to z \in y]$"*

Translating and simplifying this with $\Box$ collapsing yields: $\Box[\mathscr{V}(\vec{V}_0) \wedge \vec{V}_1 \geq_{x} \vec{V}_0 \to \Diamond_{\vec{V}_1}[\vec{V}_2 \geq_{y} \vec{V}_1 \wedge \Box_{\vec{V}_2}(\vec{V}_3 \geq_{z} \vec{V}_2 \to \Box_{\vec{V}_3}[V_4 \geq_{w} \vec{V}_3 \to (f_4(w) \in_4 f_4(z) \to f_4(w) \in_4 f_4(x))] \to f_3(z) \in_3 f_3(y)])]$.

This intuitively says that for any initial segment and assignment V_1, f_1 we can have an extending $\vec{V}_2 \geq_{y} V_1$ which assigns $f_2(y)$ to the powerset of $f_1(x)$ (where the latter notion is understood in a modal sense).[12]

Proof. Consider an arbitrary situation in which $\mathscr{V}(\vec{V}_0) \wedge \vec{V}_1 \geq_{\mathbf{x}} \vec{V}_0$. As before, we know by $\mathscr{V}(\vec{V}_1)$ and the One More Layer of Classes Lemma**??** we can have a V_2 which contains a set_2 whose elements are exactly the set_1s such that $[(\forall a)(a \in_1 b \to b \in_1 f_1(x))]$[13]. By making this choice for $f_2(y)$, we can have: $\Diamond_{\vec{V}_1}(\vec{V}_2 \geq_{\mathbf{y}} \vec{V}_1 \wedge V_2$ contains a single layer of classes over $\vec{V}_1 \wedge f_2(y)$ contains all subsets of $f_1(x)$ in the sense of V_1).

Entering this $\Diamond$ scenario, we can deduce that $f_2(y)$ also contains all subsets of $f_2(x)$ *in the sense of* V_2, i.e., $\Box_{\vec{V}_2}[(\forall k)(C(k) \to k \in_2 f_2(x)) \to (\exists k')(k' \in_2 f_2(y) \wedge (\forall k)[C(k) \leftrightarrow k \in_2 k'])]$, ([review wording] proving this fact will be helpful, because it is context-restricted to V_2, hence can be imported into into any context where the V_2 facts are held fixed.) .[14]. In-

[12] Specifically, if any extending $\vec{V}_3 \geq_{\mathbf{z}} \vec{V}_2$ which assigns $f_3(z)$ to something that behaves like a subset of x (in the sense that any $\vec{V}_4 \geq_{\mathbf{w}} \vec{V}_3$ must satisfy $f_4(w) \in_4 f_4(z) \to f_4(w) \in_4 f_4(x)$) must satisfy $f_3(z) \in_3 f_3(y)$.

[13] Specifically, by Simple Comprehension it's possible that the otherwise unused predicate H applies to exactly those a such that $set_1(a) \wedge [(\exists b)(a \in_1 b \wedge b \in_1 f_1(x))$. So by our characterization of V_2 as containing one more layer of sets there is a unique k which contains all and only the a satisfying the condition above.

[14] To show this, consider an arbitrary situation (holding $\vec{V}_1, \vec{V}_2$ fixed) in which $(\forall k)(C(k) \to k \in_2 f_2(x))$. By the fact that $\vec{V}_2 \geq_{\mathbf{y}} \vec{V}_1$ we have $(\forall k)(k \in_2 f_2(x) \to k \in_1 f_1(x))$. Then every object satisfying C is available at a level below the level where $f_2(x) = f_1(x)$ first occurs.

formally, this says: it's logically necessary (given the facts about V_2) that if C only applies to objects in $f_2(x)$ then there is some set_2 in $f_2(y)$ which has exactly the objects satisfying C as elements.

Now, it remains to consider an arbitrary situation (holding the facts about our $\vec{V}_2$ fixed) in which $\vec{V}_3 \geq_{\mathbf{z}} \vec{V}_2 \wedge \Box_{\vec{V}_3}[V_4 \geq_{\mathbf{w}} \vec{V}_3 \rightarrow (f_4(w) \in_4 f_4(z) \rightarrow f_4(w) \in_4 f_4(x))]$ (call this hypothesis α) and show that $f_3(z) \in_3 f_2(y)$. From the $\Box_{\vec{V}_3}$ claim in our hypothesis, we can deduce that $f_3(z)$ is a subset of $f_3(x)$ in the sense of V_3 [15]. And because $\vec{V}_3 \geq_{\mathbf{z}} \vec{V}_2$, we can further deduce that everything which is $\in_3 f_3(z)$ is also $\in_2 f_2(x)$ [16]. In this way, the elements of $f_3(z)$ correspond to a logically possible subset of $f_2(x)$.

Since any V_3 extending V_2 can't add any elements to $f_2(x)$, it is straightforward to verify that $f_3(z) \in_3 f_3(y)$.[17]

Thus there is a set_1 with exactly these elements, call it k. Now by our characterization of $f_2(y)$ as containing exactly those k' such that $set_1(k') \wedge [(\forall k'')(k'' \in_1 k' \rightarrow k'' \in_1 f_1(x))$ we can deduce that $k' \in_2 f_2(y)$. Thus we have $(\exists k')(k' \in_2 f_2(y) \wedge (\forall k)[C(k) \leftrightarrow k \in_2 k'])$ as desired.

[15] [this is way too long...you just should say something like..this follows by blah] Suppose for contradiction that $(\exists k'')(k'' \in_3 f_3(z) \wedge \neg k'' \in_3 f_3(x))$. Then (by simplified choice and various applications of simple comprehension combined as per the Multiple Definitions Lemma) it is $\Diamond_{\vec{V}_3}$ that $V_4 \geq_{\mathbf{w}} \vec{V}_3$ with $V_4 =_{set} V_3$ and $f_4(w)$ applies to a unique object k'' such that k" $\in_3 f_3(z) \wedge \neg k'' \in_3 f_3(x)$. In this $\Diamond_{\vec{V}_3}$ context, it must also be true that that $f_4(w) \in_4 f_4(z) \wedge \neg f_4(w) \in_4 f_4(x)$.

But the logical possibility of such a scenario (holding fixed the facts about $\vec{V}_3$) contradicts our prior assumption $\Box_{\vec{V}_3}[V_4 \geq_{\mathbf{w}} \vec{V}_3 \rightarrow (f_4(w) \in_4 f_4(z) \rightarrow f_4(w) \in_4 f_4(x))]$.

Thus $f_3(z)$ is a subset of $f_3(x)$ (in the sense of V_3).

[16] Any $k \in_2 f_3(z)$ must also be in $\in_3 f_3(z)$ hence in $\in_3 f_3(x) = f_2(x)$ hence $\in_2 f_2(x)$

[17] We can import the fact that $f_2(y)$ contains all logically possible subsets of $f_2(x)$ in the sense of V_2, since this claim is content restricted to $\vec{V}_2$. By Simple Comprehension, it is possible (holding fixed the facts about $\vec{V}_2, \vec{V}_3$ and hence the facts about $f_2(y)$ and $f_3(z)$ just proved above) for C to apply to exactly the elements of $\in_3 f_3(z)$. Because $(\forall k)[k \in_3 f_3(z) \rightarrow k \in_2 f_2(x)])$ remains true in this context, it follows that that C only applies to objects $\in_2 f_2(x)$. Because our characterization of $f_2(y)$ remains true in this context (note that it is content restricted to $\vec{V}_2$), we know that (necessarily, given V_2 facts) if C only applies to objects $\in_2 f_2(x)$ there is some set_2, k, such that $k \in_2 f_2(y)$ whose elements are exactly the objects satisfying C. Thus there is some set_2 which has exactly the same elements as $f_3(z)$. Now, by the thinness/extensionality requirement built into

From this proof of $\Box_{V_2}(\alpha \to f_3(z) \in_3 f_3(y))$, the desired conclusion follows straightforwardly. $\Box$

Proposition 14.1.7 (Choice)**.** "$\forall x \left[\varnothing \notin x \to \exists f : x \to \bigcup x \quad \forall a \in x\, (f(a) \in x)\right]$"

Writing out almost all the abbreviations, and applying FOL to this yields:
$(\forall x)[(\forall y)(y \in x \to (\exists z)(z \in y)) \to$

$(\exists f)(\forall a)[a \in x \to \exists y(\langle a, y\rangle \in f \wedge (\forall y')[\langle a, y'\rangle \in f \to y = y'])]]$

thus it gets translated as something with the following form [inc: footnote that I'm not expanding out the brackets for pairing?]:

[fix linebreaks]

$\Box[\mathscr{V}(\vec{V}_0) \wedge V_1 \geq_x \vec{V}_0 \to [$

$\Box_{\vec{V}_1}(V_2 \geq_y \vec{V}_1 \wedge f_2(y) \in_2 f_2(x) \to \Diamond_{\vec{V}_2}[V_3 \geq_z \vec{V}_2 \wedge f(z) \in f(y)]) \to$

$\Diamond_{\vec{V}_1}(V_2 \geq_f \vec{V}_1 \wedge t_2((\forall a)[a \in x \to \exists y(\langle a, y\rangle \in f \wedge (\forall y')[\langle a, y'\rangle \in f \to$
$y = y'])]))$

So it says: if V_1 assigns $f_1(x)$ to something which doesn't contain the empty set[18]*, then one can have an extending V_2, f_2 which assigns $f_2(f)$ to a set which codes up a choice function for $f_1(x)$*[19]*.*

Proof. Unsurprisingly, we will use an instance of the Choice Axiom Schema to prove this claim.

Consider an arbitrary situation in which $\mathscr{V}(\vec{V}_0) \wedge \vec{V}_1 \geq_{\mathbf{x}} \vec{V}_0$.

Now suppose that the antecedent of the conditional we need to prove. That is, suppose that $\Box_{\vec{V}_1}(V_2 \geq_{\mathbf{y}} \vec{V}_1 \wedge f_2(y) \in_2 f_2(x) \to \Diamond_{\vec{V}_2}[\vec{V}_3 \geq_{\mathbf{z}} \vec{V}_2 \wedge f(z) \in$

$\vec{V}_3 \geq_z V_2$, we know that $f_3(z) =$ this set_2, so we have $f_3(z) \in_2 f_2(y)$, and hence $f_3(z) \in_3 f_3(y)$.

[18]in the sense that for any extending V_2, f_2 assigning y to something in $f_1(x)$ there could be a V_3, f_3 assigning z to something in $f_2(y)$)

[19]in the sense that for any extension V_3, f_3 assigning a to something in $f_1(x)$ makes $t_3(\exists y(\langle a, y\rangle \in f \wedge (\forall y')[\langle a, y'\rangle \in f \to y = y']))$ come out true

$f(y)$]).

Our first step will be to deduce from the above assumption that the empty set is not an element of $f_1(x)$, (i.e. $(\forall k)k \in_1 f_1(x) \to \exists k' k' \in_1 k$). We will argue by contradiction.

If an empty set were in $f_1(x)$, then it would be possible (holding fixed the facts about V_1) to have an extending V_2 where $f_2(y)$ is this empty set, hence it is impossible for there to be an extending V_3 where $f_3(z) \in_3 f_3(y)$. But this contradicts the $\Box_{V_1}$ assumption above. [20]

Thus we know that the empty set is not in $f_1(x)$. Now we will (unsurprisingly!) use the Choice Axiom in my formal system to construct a suitable logically possible V_2, f_2, and then show it behaves as desired. By three applications of the One More Layer Lemma, we can have a V_2 which adds three layers of classes to V_1. By Simple Comprehension, it is possible to have an index property $\boldsymbol{I}$ apply to exactly the elements of $f_1(x)$ and a relation $\boldsymbol{R}$ (which we intend to apply Choice to) which applies to exactly pairs $\boldsymbol{a}, \boldsymbol{b}$ consisting of an element $\boldsymbol{a} \in_1 f_1(x)$ and $\boldsymbol{b} \in_1 \boldsymbol{a}$. By an application of Choice to $\boldsymbol{R}$ (importing the fact that $f_1(x)$ does not contain an empty set), we can conclude it is possible that $\hat{\boldsymbol{R}}(\boldsymbol{a}, \boldsymbol{b})$ associates each $\boldsymbol{a}$ in $f_1(x)$ with a unique $\boldsymbol{b}$ in $\boldsymbol{a}$. By the Multiple Definitions Lemma we can put all these stipulations together, and then enter a single $\Diamond_{V_1}$ context in which all the characterizations of $V_2, \boldsymbol{I}, \boldsymbol{R}, \hat{\boldsymbol{R}}$ above remain true.

[20]More pediantically, suppose an empty set were in $f_1(x)$. Then it would be $\Diamond_{V_1}$ to have $\vec{V_2} \geq_y \vec{V_1}$, where $V_1 =_{set} V_2$ and $f_2(y)$ is the empty set (in the sense of $\in_1$). Since $f_2(y) \in_1 f_1(x)$, we have $set_1(f_2(y))$ and hence this $f_2(y)$ is an empty set in the sense of $\in_2$ as well, i.e., $\neg(\exists k)(k \in_2 f_2(y))$. So we also have $\neg\Diamond_{\vec{V_2}}[V_3 \geq_{\mathbf{z}} \vec{V_2} \wedge f_3(z) \in_3 f_3(y)]$, since any such $f_3(z) \in_3 f_3(y) = f_2(y)$ would have to be $\in_2 f_2(y)$. Thus we get the possibility of a scenario is ruled out by the $\Box_{V_1}$ assumption above.

Now we can show (laboriously but straightforwardly) that this V_2 contains a set_2 which is the graph of the the choice function $\hat{R}$ specified above. (With a suitable use of Wrapping Trick to mimic $\forall I$ arguments involving modality) we can note that for each $b \in_2 f_1(x)$ there is a c such that $\hat{R}(b,c)$ within V_1. Then we can exploit the fact that V_2 contains three layers of classes over V_1 to show that it contains a pair set $\langle b,c\rangle$, and a set_2 which collects together all such pairs.[21]

Finally, it remains to check that this assignment $f_2(f)$ ensures the truth of $t_2((\forall a)[a \in x \rightarrow \exists y(\langle a,y\rangle \in f \wedge (\forall y')[\langle a,y'\rangle \in f \rightarrow y = y'])])$. This is somewhat laborious, but we can do it via exactly the same technique demonstrated in the simpler proofs above. Specifically, we argue that all extending V_i, f_i which satisfy relevant antecedents must assign variables to objects at or below $f_2(f)$ and/or $f_2(x)$ (hence to objects in V_2), and then exploit the fact that $f_2(f)$ is the graph of a choice function for $f_2(x)$ [in the sense restricted to V_2]. This completes our Inn $\Diamond$ argument that a suitable extending V_2, f_2 is possible. $\square$

[21] More pedantically: by our characterization of V_2, there is (one layer above V_1) a $w = \{c\}, w' = \{b\}$ $w'' = \{b,c\}$, and hence (two layers above V_1) a $w'' = \langle b,c\rangle$. Since this is true for each b in the domain of $\hat{R}$, there will be (three layers above V_1) a set_2 which is the graph of $\hat{R}$ i.e., $\{\langle b,c\rangle$ such that $\hat{R}(b,c)\}$.Consider applying Simple Choice to specify the application a property K $(\forall x)[K(x) \leftrightarrow \exists b \exists c(\hat{R}(b,c) \wedge \exists w \exists w' \exists w''\, w = \{c\} \wedge w' = \{b\} \wedge w'' = \{b,c\}]$ (with all abbreviations written out in the usual way). By the reasoning above, there will be for each b,c such that $\hat{R}(b,c)$ a corresponding element of K. Also (again, by the reasoning above) all these elements will occur below the last layer of V_2, so by our construction of V_2 there will be a set_2 whose elements are exactly those in the extension of K. By our construction of $\hat{R}$, this set_2 is the graph of a choice function for $f_1(x)$ (i.e., it contains, for each $b \in_2 f_2(x)$, exactly one set of the form $\langle b,c\rangle$, with c such that $b \in_2 c$). So, (by the multiple definitions lemma and ignoring) it is $\Diamond_{\vec{V}_1}$ to have $\vec{V}_2 \geq_f \vec{V}_1$ with $f_2(f)$ the graph of a choice function for $f_1(x) = f_2(x)$. [in the sense restricted to V_2]

14.2 Comprehension

Proposition 14.2.1. ***Comprehension*** *"Let $\phi(x, w_1, \dots, w_n)$ be a formula in the language of ZFC with free variables $x, w_1, \dots, w_n$. Then:*

$\forall z \forall w_1 \forall w_2 \dots \forall w_n \exists y \forall x[x \in y \Leftrightarrow (x \in z \wedge \phi)]$."

Translating and then applying $\Box$ simplification yields: [fix double subscripts] $\Box[\mathscr{V}(\vec{V}_0) \wedge \vec{V}_1 \geq_z \vec{V}_0 \wedge \vec{V}_2 \geq_{w_1} \vec{V}_1 \vec{V}_1 \wedge \dots V_{n+1} \geq_{w_n} V_n \to \Diamond_{V_{n+1}}[V_{n+2} \geq_y V_{n+1} \wedge \Box_{V_{n+2}}(V_{n+3} \geq_x V_{n+2} \to [f_{n+3}(x) \in_{n+3} f_{n+3}(y) \leftrightarrow f_{n+3}(x) \in_{n+3} f_{n+3}(z) \wedge t_{n+3}(\phi)])]$

This says approximately the following. Fix assignments for for $z, w_1, \dots, w_n$ from set_{n+1} within some V_{n+1}. It's logically possible to have an extending $V_{n+2} \geq_y V_{n+1}$ which assigns y to a set which collects together exactly those x in $f_{n+2}(z)$ such that that any extending V_{n+3}, f_{n+3} which assigns $f_{n+3}(x)$ to one of these x must make $t_{n+3}(\phi(z, w_1, \dots, w_n, x))$ true.

Proof. Suppose that $\vec{V}_0 ... \vec{V}_{n+1}$ are as above.

Our first task will be to establish the logical possibility of a suitable $f_{n+2}(y)$ and V_{n+2}. Let t_{n+3**} represent the result of replacing all occurrences of relations in V_{n+3}, f_{n+3} in t_{n+3}, with occurrences of relations in V_*, f_*.[22] By using the Modal Comprehension Schema, we can show that a predicate $\boldsymbol{P}$ could apply to exactly $x \in_{n+1} f_{n+1}(z)$ with the following modal property: there could be an extension $\vec{V}_{n+3}* \geq_x \vec{V}_{n+1}$ such that $f_{n+3} * (x) = x$ and $t_{n+3}**(\phi)$ comes out true (note that $t_{n+3}**(\phi)$ makes mention of $f_{n+3}*(x)$).[23]

[22] So, as in the proof of lemma **??** occurrences of set_{n+3} are replaced with occurrences of set_*, but occurrences of set_{n+4} inside $\Box$es and $\Diamond$s are unchanged.

[23] To see why this is true more formally, consider the formula asserting that it is logically possible that f_{n+3} matches f_{n+1} everywhere but on the variable x which it takes the unique value satisfying $\boldsymbol{Q}$ (where $\boldsymbol{Q}$ is the predicate from the Modal Comprehension axiom) and

These will turn out to be exactly the objects we want our set $f_{n+2}(y)$ to collect. By the fact that $\mathscr{V}(V_{n+1})$, there's a set_{n+1} y, whose elements are exactly those those satisfying $\boldsymbol{P}$. This will be our choice for $f_{n+2}(y)$ and we will let V_{n+2} be equal to V_{n+1}.

Now it remains to check that V_{n+2}, f_{n+2} behaves as desired. We need to show that $\Box_{V_{n+2}}$ if $\vec{V_3} \geq_x \vec{V_2}$, assigns $f_{n+3}(x)$ to something in $f_{n+3}(y)$ iff it satisfies $t_{n+3}(\phi(z, w_1, \dots, w_n, x))$. By Ign we know that if there could be a counterexample to the claim above, then there could be a counterexample which holds fixed $V_{n+1}, \boldsymbol{P}$ as well as V_{n+2}. So consider an arbitrary scenario (holding fixed $V_{n+1}, \boldsymbol{P}, V_{n+2}$) in which $\vec{V}_{n+3} \geq_{\mathbf{x}} \vec{V}_{n+2}$. It suffices to show that $f_{n+3}(x) \in_{n+3} f_{n+3}(y) \leftrightarrow f_{n+3}(x) \in_{n+3} f_{n+3}(z) \wedge t_{n+3}(\phi)$ in this scenario. There are two directions to check.

$\rightarrow$ Suppose f_{n+3} assigns x to something in $f_{n+3}(y)$ (our supposed comprehension set). Then our characterization of $f_{n+2}(y)$[24] implies that this object is in $f_{n+1}(z)$ (the set we are comprehending over). So we have $f_{n+3}(x) \in_{n+3} f_{n+3}(z)$ immediately. Now we need $t_{n+3}(\phi)$. Our characterization of $f_{n+2}(y)$ also says [via the wrapping trick for mimicing quantifying in] that because $f_{n+3}(x) \in_{n+2} f_{n+2}(y)$, it is possible (holding fixed V_1) for an extension $\vec{V}_{n+3}* \geq_x \vec{V}_{n+1}$ which assigns $f_{n+3} * (x) =$ to (an object in structurally the same position w.r.t. V_{n+1}, f_{n+1} as our) $f_{n+3}(x)$ to make $t_{n+3} * *(\phi)$

$t_{n+3}(\phi)$ comes out true.

$\Diamond_{\vec{V}_{n+1}, Q}[(\forall r \neq \ulcorner x \urcorner) f_{n+3}(r) = f_{n+1}(r) \wedge$
$(\forall q)(f_{n+3}(\ulcorner x \urcorner) = q \leftrightarrow Q(q)) \wedge$
$t_{n+3}(\phi)]$

This formula can be plugged directly into the Modal Comprehension axiom, and we can derive that the resulting property $\boldsymbol{P}$ applies to all and only those $x \in_{n+1} f_{n+1}(z)$ with the property informally described above.

[24] This must remain true in our current context because it is content-restricted to V_{n+1}, V_{n+2}.

true.

We can infer that the same scenario is possible while holding fixed the $\vec{V_{n+1}}, \vec{V_{n+2}}$ facts as well, by Ignoring.[25]. So we can enter this $\Diamond_{\vec{V_{n+1}},\vec{V_{n+2}},\vec{V_{n+3}}}$ context, and import all previously established facts about V_{n+2} and V_{n+3}. Now it remains to use the Translation Lemma to go from $t_{n+3} **(\phi)$ to $t_{n+3}(\phi)$.

The trick will be to cook up a $V_{n+1}, f_{n+1}*$ which agrees with t_{n+3} and $t_{n+3}**$ on the assignment of x and all other variables free in ψ, and then use a version of the Translation Lemma to go from $t_{n+3}**(\phi)$ to $t_{n+1@}(\phi)$ to $t_{n+3}(\phi)$. For, note that we have $V_{n+3} \geq V_{n+1}$ and $V_{n+3}* \geq V_{n+1}$ and that f_{n+3} agrees with $f_{n+3}*$ in assigning all variables free in ϕ to objects in V_1: y is not free in ϕ, f_{n+3} agrees with $f*_{n+3}$ on the assignment of x to something $\in f_{n+1}(x)$ hence in V_1 by construction, and on all other free variables $w_1...w_n$ in ϕ both f_{n+3} and $f_{n+3}*$ agree with f_{n+1}). Thus if we use modal comprehension to let $f_{n+1}@ = f_{n+1}$ everywhere except in assigning x to $f_{n+3}(x) = f_{n+3*}(x)$, inside this $\Diamond_{\vec{V_{n+1}},f*,\vec{V_{n+2}},\vec{V_{n+2}}}$ scenario we will have $\mathscr{V}(V_{n+1}, f_@)$. Thus we will try to use the Translation lemma to get $t_{n+3} **(\phi) \leftrightarrow t_{n+1@}(\phi) \leftrightarrow t_{n+3}(\phi)$, as desired.

Once we have done this, we are finished. For, from the fact that $t_{n+3}(\phi)$ in the above $\Diamond_{\vec{V_{n+1}},f*,\vec{V_{n+2}},\vec{V_{n+3}}}$ scenario, we can infer that it holds in our original scenario as well.

[Now it just remains to deal with the wrinkle that (as before) the Translation Lemma doesn't directly say anything about $t_{n+3} **(\phi)$ or $t_{n+1@}(\phi)$.

[25] This inference is permitted because the inside of the $\Diamond_{\vec{V_{n+1}}}$ claim is content restricted to $\vec{V_{n+1}}, \vec{V_{n+3}}*$ and there is no overlap between $V_{n+3}*$ and V_{n+2}, V_{n+3}

However, we can use the $\Box$ relabling to get what we need as before. First replace all instances of f_{n+1} with $f_{n+1@}$. Then replace all instances of V_{n+3}, f_{n+3} with corresponding V*,f*, but notice that there may be some collateral damage. Any mentions of V_{n+3} within $t_{n+1}(\phi)$ will be replaced, so we have more work to do if ϕ it contains any quantifiers nested 2 deep. Fortunately, however, we can undo this damage, by entering into the t_{n+2} contexts housing each instance of such nested quantification which got changed. The $\Box$ and $\Diamond$ relabeling let us derive that $\Box t_{n+2}(\rho) \leftrightarrow t_{n+2}(\rho)[t_{n+3}/t_{n+3} **]$ (or the corresponding $\Diamond$ claim in each of these contexts, and hence to fix all such collateral damage.] [FIX wording]

$\leftarrow$ Conversely, suppose f_{n+3} assigns x to something in $f_{n+3}(z)$ (the set being comprehended over) and that $t_{n+3}(\phi)$. By our characterization of $f_{n+2}(y)$, we can show that the relevant object is also in $f_{n+3}(y)$ if we establish two things. First, we need the object is $\in_1 f_{n+1}(z)$. This follows immediately, because $\vec{V_{n+3}} \geq_x \vec{V_{n+2}} \geq_y \vec{V_{n+1}}$.

Second, we need to show that it is $\Diamond_{V_{n+1}}$ to have $\vec{V_{n+3}}* \geq_x \vec{V_{n+1}}$ such that [again, speaking loosely and using the Wrapping Trick to mimic quantifying in] $f_{n+3} * (x) =$ this $f_{n+3}(x)$ and $t_{n+3} * *(\phi)$.

I will prove this by proving the stronger corresponding $\Diamond_{\vec{V_{n+1}},\vec{V_{n+2}},\vec{V_{n+3}}}$, claim. By assumption, we have $t_{n+3}(\phi)$. By simple comprehension, it is $\Diamond_{\vec{V_{n+1}},\vec{V_{n+2}},\vec{V_{n+3}}}$ to have $t_{n+3}(\phi)$ remain true while $V_{n+3}* =_{set} V_{n+3}$ and $f_{n+3}*$ agrees with f_{n+3} everywhere, except that $f_{n+3}(y) = f_{n+1}(y)$ it agrees with f_{n+1} on the assignment of y. Now (just as above) we can use the generalized Translation Lemma to go from the fact that V_{n+3} and this V_{n+3*} both extend V_{n+1} and agree in assigning all variables free in ϕ (because y is not free in

ϕ) to objects in V_{n+1} to the conclusion that $t_{n+3}(\phi) \leftrightarrow t_{n+3} **(\phi)$. This gives us $t_{n+3}(\phi)$, as desired.

Combining the $\rightarrow$ and $\leftarrow$ arguments above complete the desired proof that $f_{n+3}(x) \in f_{n+3}(y) \leftrightarrow f_{n+3}(x) \in_{n+3} f_{n+3}(z) \wedge t_{n+3}(\phi)$.

$\square$

14.3 Infinity

Proposition 14.3.1. ***Infinity***"$\exists x\, [\varnothing \in x \wedge \forall y(y \in x \rightarrow S(y) \in x)]$ *."*

where $S(x)$ *is* $x \cup \{x\}$*.*

Let

$$\ulcorner \varnothing \in f_1(x) \urcorner = \Diamond_{\vec{V}_1}(\vec{V}_2 \geq_e \vec{V}_1 \wedge \Box_{\vec{V}_2}[\vec{V}_3 \geq_z \vec{V}_2 \rightarrow \neg f_3(z) \in_3 f_3(e)] \wedge f_2(e) \in_2 f_2(x))$$

$$\ulcorner S(f_2(y)) \in f_2(x) \urcorner = \Diamond_{\vec{V}_2}(\vec{V}_3 \geq_s \vec{V}_2 \wedge \Box_{\vec{V}_3}[V_4 \geq_z \vec{V}_3 \rightarrow f_4(z) \in_4 f_4(s) \leftrightarrow f_4(z) \in f_4(y) \vee f_4(z) = f_4(y)] \wedge f_3(s) \in_2 f_3(x)])$$

Using these suggestively named components, the translation of infinity can be written as:

$$\Box(\mathscr{V}(V_0) \rightarrow \Diamond_{\vec{V}_0}[\vec{V}_1 \geq_x \vec{V}_0 \wedge \ulcorner \varnothing \in f_1(x) \urcorner$$

$$\wedge \Box_{\vec{V}_1}(\vec{V}_2 \geq_y \vec{V}_1 \wedge f_2(y) \in_2 f_2(x) \rightarrow \ulcorner S(f_2(y)) \in f_2(x) \urcorner)]$$

Proof. Consider an arbitrary scenario in which $\mathscr{V}(V_0)$ holds. On this assumption, we can show that the suggestive names we used for parts of the translation above are accurate: if $\vec{V}_1 \geq_{\mathbf{x}} \vec{V}_0$ then $\ulcorner \varnothing \in f_1(x) \urcorner$ holds if $\varnothing \in_1 f_1(x)$ and if, furthermore, $\vec{V}_2 \geq_{\mathbf{y}} \vec{V}_1$, then $\ulcorner S(f_2(y)) \in f_2(x) \urcorner$ holds if

$S(f_2(y)) \in_2 f_2(x)$.

For example, note that if $\varnothing \in_1 f_1(x)$ then it's possible to have $V_2 =_{set} V_1$ and f_2 equal to f_1 everywhere except at e and $f_2(e) = \varnothing$. It's thus necessary, holding V_2, f_2 fixed, that if $\vec{V_3} \geq_{\mathbf{z}} \vec{V_2}$ then $\neg f_3(z) \in_3 f_3(e)$ as $f_3(e) = f_2(e) = \varnothing$. This establishes the claim about $\ulcorner \varnothing \in f_1(x) \urcorner$. Similar elementary reasoning establishes the above claim about $\ulcorner S(f_2(y)) \in f_2(x) \urcorner$.

I will establish the logical possibility claim that we need, by arguing as follows. By the Infinite Well-Ordering Lemma (proved in section 12.0.2) there can be a an infinite well ordering $\omega, \leq$ which contains only successor stages. By the Fleshing Out Lemma (C.7), it is logically possible to have an initial segment V_ω, whose ordinals $ord_\omega, \leq_\omega$ are isomorphic to $\omega, \leq$. Using the Recursive Definition Lemma (proved in section B.1.1), we define a function F from ω to V_ω with $F(0) = \varnothing$ and $F(n+1) = S(F(n))$ and then use induction establish the domain of F is ω. By the definition of ω, for each $n \in \omega$ there is an $n + 1 \in \omega$ such that $F(n + 1) = S(F(n))$. We then establish the possibility of an initial segment $V_{\omega+1}$ containing an extra layer of sets over those in V_ω and thus containing a set x whose members are exactly the elements in the range of F. The theorem follows by observing that letting V_1 be $V_{\omega+1}$ makes the sentence true.

Now let us go into details. By the Infinite Well-Ordering Lemma, we can have a well-ordering $\omega, \prec$ without a maximal element where every element satisfying ω is either 0 or a successor.

By the Fleshing Out Lemma we can infer $\Diamond_{\omega,\prec} \mathscr{V}(\text{set}_\omega, \in_\omega, @_\omega, \omega, \prec)$. Assume V_ω is the tuple of relations having these properties. Next we can use the Recursive Definition Lemma to establish the logical possibility of a two place relation $F(o, z)$ between objects satisfying ω and set$_\omega$ adopting the functional abbreviation $F(o) = z$ for clarity

$$F(o) = z \iff \begin{cases} o = 0 \wedge z = \varnothing \\ \qquad \vee \\ o = n + 1 \wedge z = F(n) \cup \{F(n)\} \end{cases}$$

Where $\varnothing$ is the element in set$_\omega$ containing no other elements under $\in_\omega$ and $F(n) \cup \{F(n)\}$ is the element in set$_\omega$ whose elements are exactly the members of $F(n)$ and $F(n)$.

[We can check that the premises needed for the Recursive Definition Lemma are satisfied, as follows].Clearly, it is logically necessary (given the facts about $\omega, \leq$ and V_ω) that a unique object satisfies $x = \varnothing$ in V_ω. And for n s.t. $\neg n = 0$ and $\omega(n)$, we know that n is a successor ordinal (so there is an m such t $n = m + 1$) by our characterization of $\omega, \leq$. Thus [it is logically necessary (given the facts about $\omega, \leq$ and V_ω) that] if F is defined and functional below n, we have the existence of an x such that $(\exists m)[n = m + 1 \wedge x = S(F(m))]$ because ord_ω include a successor ordinal for every ordinal which it contains (and hence a stage above every stage it contains) and $S(F(m))$ must occur a stage above wherever $F(m)$ occurs (by the fact that $\mathscr{V}(V_\omega)$ and our definition S). We the have uniqueness of this x by the extensionality of the set_ω and the definition of S.

Now by the One More Layer Lemma (proved in section C.4) we can infer the possibility of $V_{\omega+1}$ extending V_ω and adding a single layer of classes. Now all the objects in the image of F are sets in V_ω. Thus $V_{\omega+1}$ contains a set I

whose members are exactly those elements of V_ω such that $(\exists o)(\omega(o) \wedge F(o) = x)$. This set contains $\varnothing$ (a set which has no elements in the sense of $V_{\omega+1}$ and hence also none in the sense relevant to $V_{\omega+2}$) and is closed under application of $\boldsymbol{S}$.

Lastly, it remains to show that we can find a set like $\boldsymbol{I}$ in an initial segment extending $\vec{V}_0$. By the Hierarchy Extending Lemma (proved in C.5) if is logically possible to have an extension V_1 of $\vec{V}_0$, such that $\boldsymbol{Z}$ isomorphically maps from $V_{\omega+1}$ to an initial segment of V_1. It is a straightforward, if somewhat tedious, process to verify that the image of our $\boldsymbol{I}$ under $\boldsymbol{Z}$ also behaves like a suitable infinite set: it contains an object $\varnothing$ which has no elements in the sense of V_1, and contains the the successor of every set_1 it contains.

To complete the proof, note that we can let $f_1(x)$ be $\boldsymbol{Z}(\boldsymbol{I})$. Clearly $\ulcorner \varnothing \in f_1(x) \urcorner$ holds in this case and if $\vec{V}_2 \geq_{\mathbf{y}} \vec{V}_1 \wedge f_2(y) \in_2 f_2(x)$ then $\boldsymbol{S}(f_2(y)) \in f_2(x)$ so $\ulcorner\ \boldsymbol{S}(f_2(y)) \in f_2(x) \urcorner$ holds.

□

14.4 Replacement

Proposition 14.4.1. ***Replacement***

"The axiom schema of replacement asserts that the image of a set under any definable function will also fall inside a set.

Formally, let ϕ be any formula in the language of ZFC whose free variables are among $x, y, A, w_1, \ldots, w_n$, so that in particular B is not free in ϕ. Then:

$$\forall A \forall w_1 \forall w_2 \ldots \forall w_n \big[\forall x(x \in A \to \exists!\, y \phi) \to \exists B \forall x \big(x \in A \to \exists y(y \in$$

$\boldsymbol{B} \wedge \phi))\big].$

In other words, if the relation ϕ *represents a definable function f, A represents its domain, and f(x) is a set for every x in that domain, then the range of f is a subset of some set B."*

Instances of this schema have a translation with the form

$\Box[\mathscr{V}(\vec{V}_0)\Box(\vec{V}_1 \geq_{\boldsymbol{a}} \vec{V}_0 \rightarrow \Box[\vec{V}_2 \geq_{\boldsymbol{w1}} \vec{V}_1 \ldots \Box V_{n+1} \geq_{\boldsymbol{wn}} V_n \rightarrow (\alpha \rightarrow \beta)]...])]$

which, by $\Box$ *simplification, becomes:*

$\Box[\mathscr{V}(\vec{V}_0) \wedge \vec{V}_1 \geq_{\boldsymbol{a}} \vec{V}_0 \wedge \vec{V}_2 \geq_{\boldsymbol{w1}} \vec{V}_1 \ldots \Box V_{n+1} \geq_{\boldsymbol{wn}} V_n \rightarrow (\alpha \rightarrow \beta)]]$

where:

- $\alpha = \Box_{V_{n+1}}(V_{n+2} \geq_{\boldsymbol{x}} V_{n+1} \wedge f_{n+2}(x) \in_{n+2} f_{n+2}(a) \rightarrow \Diamond_{V_{n+2}}[V_{n+3} \geq_{\boldsymbol{y}} V_{n+2} \wedge \boldsymbol{t}_{n+3}(\phi(w_1, \ldots w_n, x, y)) \wedge \Box_{V_{n+3}} V_{n+4} \geq_{\boldsymbol{z}} V_{n+3} \wedge \boldsymbol{t}_{n+4}(\phi(w_1, \ldots w_n, x, z)) \rightarrow f_{n+4}(y) = f_{n+4}(z))])$

- $\beta = \Diamond_{V_{n+1}}(V_{n+2} \geq_{\boldsymbol{b}} V_{n+1} \wedge \Box_{V_{n+2}}[V_{n+3} \geq_{\boldsymbol{x}} V_{n+2} \wedge f_{n+3}(x) \in_{n+3} f_{n+3}(a) \rightarrow \Diamond V_{n+4} \geq_{\boldsymbol{y}} V_{n+3} \wedge f(y) \in f(b) \wedge \boldsymbol{t}_4(\phi(w_1, \ldots w_n, x, y))])$

Proof Sketch:

In essence, the translation of the Replacement Schema's antecedent $[\alpha]$ asserts that for every x in a there is a logically possible [it would be possible to have an] initial segment V_x and an element y of that segment such that y is the unique solution to $t(\phi(x, y))$.

And the translation of Replacement's consequent $[\beta]$ demands that we produce a *single* logically possible initial segment [(call it V_Σ)] containing a y for every x in a (technically containing a set b containing all such y's but that is fixed by one more layer)[satisfying $t_\Sigma(\phi(x, y))$.

Now, the Translation Lemma tells us that if $t_x(\phi(x, y))$ holds in some V_x, then it holds in any extension of V_x which preserves the assignment of x and y and all the other free variables in ϕ. Thus, it is enough to demonstrate the possibility of some V_Σ extending each V_x.

To achieve this end, we first invoke Combinatorial Replacement to [get (the logical possibility of) simultanioulsy having a collection of hierarchies V_x parametrized to each $x \in_n f_n(a)$] parameterized the V_x by x and then invoke the Mass Hierarchy Combining Lemma (proved in C.6) to (essentially) get a single initial segment extending them all. Adding one extra layer of sets on top of that is enough to produce the desired set $\boldsymbol{B}$.

Proof. Consider an arbitrary situation with $\vec{V_0}...\vec{V_{n+1}}$ as above. Assume that our translation of the antecedent to replacement, α, is true.

Constructing the $V_x s$ with Combinatorial Replacement

[fill in missing "vec"s as per new notion]

Our first step will be to use the Combinatorial Replacement Schema to establish that a single scenario could associate each $x \in_{n+1} f_{n+1}(a)$ with a corresponding initial segment V_x extending V_{n+1} and containing a witness y satisfying $t(\phi(x, y))$.

Our assumption α guarantees that for any $\vec{V_{n+2}}$ extending V_{n+1} which assigns x so as to satisfy $t_{n+2}(f(x) \in f(a))$, there can be a $\vec{V_{n+3}}$ extending $\vec{V_{n+2}}$ and which assigns y so that $t_{n+3}(\phi)$ comes out true (where $t_{n+3}(\phi)$ implicitly refers to x and y via f_{n+3}).

It is logically possible that I applies to exactly those objects which are

$\in_{n+1} f_{n+1}(a)$. Entering this $\Diamond_{V^{-}_{n+1}}$ scenario, α will remain true. And it is easy to see that α implies the following modal claim. For any way P could 'select' a single object satisfying I (and hence for every possible choice of $f_{n+2}(x)$ on which $t_{n+2}((f(x) \in f(a))$ comes out true), there could be an extension V_x which agrees with V_{n+1} on everything but x and y, assigns x to the object selected by P and makes $t_x(\phi)$ come out true.

$$\Box_{V_{n+1}}(\exists!\, x P(x)) \wedge I(x) \rightarrow \Diamond_{V_{n+1},P}[V_x \geq_{x,y} V_{n+1} \wedge (\forall k)(f_x(x) = k \rightarrow P(k)) \wedge t(\phi)]$$

This statement is in the form needed to apply the Combinatorial Replacement Axiom Schema. Thus, by instantiating this schema we can derive the corresponding consequent that it is logically possible (holding fixed I, V_{n+1}) for there to simultaniously be a bunch of different $V_{\hat{x}}$ indexed to each of the different objects $\hat{x}$ satisfying I, i.e., to the $\hat{x} \in_{n+1} f_{n+1}(a)$. [More strictly we get that it is possible for there to be a relation [fill in good notation for it here] that codes up the behavior of each $V_{\hat{x}}$]

Constructing V_{n+2}, f_{n+2}

Next we want to argue that one can have an extending V_{n+2} which assigns b to an object that 'gathers up', for each possible assignment of x to something $\hat{x} \in_{n+1} f_{n+1}(a)$, (the images under isomorphism of) the choice for y made by the corresponding $V_{\hat{x}}$ in which $t(\phi(x, y))$ come out true.

First we build a suitable hierarchy of sets. We use the V-Combining Lemma to get a hierarchy of sets V_{Σ}, which has initial segments isomorphic to each of the scattered $V_{\hat{x}}$ described above (under a certain relation Z [check that def of iso only requires that Z behave like an iso when restricted to the relevant pair of objects]). Then we use One More Layer to argue for the logical possibility of extending this hierarchy of sets by one more layer. Finally we use the Hierarchy Extending Lemma to get that this structure is isomorphic to one that extends V_{n+1}.

This structure will be the V_{n+2} in our desired V_{n+2}, f_{n+2}.[26] It contains a set_{n+2} which collects together the set_{n+2} which are in the images of each $f(y)$ chosen by the V_x for $x \in_{n+1} f_{n+1}(a)$ (under the relevant combination of isomorphisms).[27] Thus we can have $\vec{V_{n+2}} \geq_b \vec{V_{n+1}}$ with $f_{n+2}(b)$ as above.

Checking that V_{n+2}, f_{n+2} behaves as intended

Finally, we must show that the V_{n+2}, f_{n+2} we have constructed makes β, the translation of the consequent of the replacement axiom schema true. Consider an arbitrary extension V_{n+3} which assigns x to something $\in_{n+2}$

[26] By the V-Combining Lemma, it is logically possible to have a V_Σ, such that each of the hierarchies of objects satisfying $set*_{n+3}(\cdot, k), \in *_{n+3}(\cdot,\cdot,k)$ for some $k \in_{n+1} f_{n+1}(a)$ is isomorphic to an initial segment of this V_Σ via the relation Z. By the Hierarchy Extending Lemma, we could have a $V_{\Sigma *} \geq_{set} V_{n+1}$, such that V_Σ is isomorphic to an initial segment of $V_\Sigma *$ via the relation Z'. Finally by the One More Layer lemma it is possible to have $V_{n+2} \geq_{set} V_{\Sigma *}$ which adds one more layer of sets to $V_{\Sigma *}$.

[27] Specifically we define $f_{n+2}(b)$ as follows:

For each $k \in_{n+1} f_{n+1}(a)$ there is a $k' = f_{3,k} * (y)$ the choice of $f_{n+3}(y)$ within the initial segment associated with k. We want $f_{n+2}(b)$ to be a set which gathers up (the isomorphic images of) all such sets. Specifically, note that each k' above gets taken to something in V_Σ by Z and then to something in $V_\Sigma *$ by Z'. By simple comprehension a property P could apply to exactly those $k*$ in $V_{\Sigma *}$ such that $\exists k \exists k' Z'(Z(f_{3,k} * (y))) = k*$. So by the fact that the sets for our V_{n+2} are generated by adding one more layer of classes to $V_{\Sigma *}$, we know that there is a set_2 with the above property, i.e., a set_{n+2} whose elements are exactly those $k*$ such that $\exists k \exists k' Z'(Z(f_{3,k} * (y))) = k*$. Let $f_{n+2}(b)$ be this set, and otherwise let $f_{n+2} = f_{n+1}$, so that we have $V_{n+2} \geq_b V_n + 1$.

Finally by the Multiple Stipulations Lemma, it is $\Diamond_{\vec{V_{n+1}}}$ to simultaniously have $\vec{V_{n+2}}, \vec{V_\Sigma}, \vec{V_\Sigma} *, Z, Z'$ satisfying all of the successive definitions above.

$f_{n+2}(a)$. We need to show that there can be an extending V_{n+4} which assigns y to something in $f_{n+2}(b)$ and satisfies $t_{n+4}(\phi)$.

To do this, we note that we must also have $x \in_{n+1} f_{n+1}(a)$,[28] hence there is some $V_{\hat{x}}$ indexed by x. This $V_{\hat{x}}$ assigns x to $f_3(x)$ and assigns y in such a way as to make $t_{\hat{x}}(\phi(x, y))$ [i.e.$t_{n+3@@}(\phi(x, y))$ in the logically possible scenario where $V_@, f_@$ behaves like $V_{\hat{x}}, f_{\hat{x}}$] true. And this $V_{\hat{x}}$ can be isomorphically mapped to an initial segment of V_{n+2} (by composing the sequence of isomorphisms mentioned above). Thus we can have $\vec{V_{n+4}} \geq_{\mathbf{y}} \vec{V_{n+3}}$ where $V_{n+4} =_{set} V_{n+3}$ and $f_{n+4}(y)$ is the image of $f_{\hat{x}}(y)[f_@]$ under this isomorphism. This choice of $f_{n+4}(y)$ immediately ensures that $t_{n+4}(y \in b)$ is true, by our characterization of $f_{n+3}(b)$.

Furthermore, there is an obvious extended isomorphism between some V_*, f_* (where V_* is an initial segment of V_{n+4}) and $V_{\hat{x}}, f_{\hat{x}}$ [i.e. $V_@, f_@$][29]. Thus by the Isomorphism Lemma we can infer from the fact that $t(\phi(x, y))$ is true in $V_{\hat{x}}, f_{\hat{x}}$ [i.e. the fact that $t_{n+3@@}(\phi(x, y))$] to the claim that it is true in V_*, f_*.[i.e., $t_{n+3**}(\phi(x, y))$]

Finally, we can use (a version of) the Translation Lemma to infer from the truth of $t(\phi(x, y))$ in V_*, f_* [i.e., $t_{n+3**}(\phi(x, y))$] to its truth in V_{n+4}, f_{n+4}. For we have $V_{n+4} \geq V_*$, and we know that $f_{n+4} = f_*$ on all variables free in ϕ as follows. On $w_1 ... w_n$, f_{n+4} agrees with f_{n+1} and so does f_*, by the fact that all the $V_{\hat{x}}, f_{\hat{x}}$ agree with V_{n+1} on these values, and some reasoning involving the Isomorphism Agreement Lemma[30]. On x, we have $f_*(x) = f_{\hat{x}}(x)[=$

[28] since $V_{n+2} \geq V_{n+1}$

[29] The only issue is to blend the isomorphism between hierarchies with the possible isomorphism between different copies of structures satisfying $PA_\Diamond$. (Note that the categoricity of $PA_\Diamond$ is an immediate correlary of the well ordering comprability lemma)

[30] Consider the isomorphism between initial segments of V_{n+4} induced by restricting the

$f_@(x)] = f_{n+4}(x)$, by our choice of which $V_{\hat{x}}$ to consider. And on y [the giant image set we have so arduously constructed] we have $f_*(y) = f_{n+4}(y)$= the isomorpic image of $f_{\hat{x}}(y)$, by our characterizations of $f_{n+4}(y)$ and f_*. Thus applying a version of the translation lemma will let us infer from truth of $t(\phi(x, y))$ in V_*, f_* [i.e., $t_{n+3**}(\phi(x, y))$] to the conclusion that $t_{n+4}(\phi(x, y))$ in the scenario above, as desired.

[The only wrinkle is that, as in previous cases, the Translation Lemma only directly tells us that $\vdash V_{n+4} \geq V_{n+3} \wedge f_{n+1}(v) = f_{n+3}(v) \wedge ... \rightarrow (t_{n+4}(\psi) \leftrightarrow t_{n+3}(\psi))$ claim. But, because of the box introduction rule, we also have $\vdash \Box()$ of the claim above. So by applying $\Box$ relabling, we can make the needed substitutions to get $\vdash \Box(V_{n+4} \geq V_{n+3**} \wedge f_{n+1}(v) = f_{n+3*}(v) \wedge ... \rightarrow (t_{n+4}(\psi) \leftrightarrow t_{n+3**}(\psi)))$. Finally, inferring from necessity to truth gives us the desired claim.]

Leaving $\Diamond$ contexts and dropping subscripts as needed gives us $\Diamond_{\vec{V}_{n+3}} \vec{V}_{n+4} \geq_{\mathbf{y}} \vec{V}_{n+3} \wedge f(y) \in f(b) \wedge t_{n+4}(\phi(x, y))$ and then β itself.

This gives us the conditional $\alpha \rightarrow \beta$, as desired. Now successively completing $\Box$I arguments and concluding conditional proofs (just as in all the previous cases) gives us the full modal translation of the relevant instance of the ZFC Replacement Schema.

$\Box$

map from $V_{\hat{x}}$ to V_* to the portion of $V_{\hat{x}}$ which is V_{n+1}. The domain of this map contains $f_{\hat{x}}(w_i)$ for each w_i, since $\vec{V}_{\hat{x}} \geq_{x,y} \vec{V}_{n+1}$. Since this map must behave the same as the identity automorphism from V_{n+1} to V_{n+1}, it must map each $f_*(w_i) = f_1(w_i)$ to $f_1(w_i)$.

Appendix A

Set Theoretic Mimicry of Sentences in $\mathcal{L}$

I will now describe how to use the familiar formal background of set theory to *mimic* intended truth conditions for statements in the a language containing the logical possibility operator $\Diamond$ alongside usual first order logical vocabulary (where distinct relation symbols R_1 and R_2 always express distinct relations) as follows.

[add rules for bot]

> A formula ψ is true relative to a model $\mathscr{M}$ and an assignment ρ which takes the free variables in ψ to elements in the domain of $\mathscr{M}$[1] just if:
>
> - $\psi = R^k_n(x_1 \ldots x_k)$ and $\mathscr{M} \models R^k_n(\rho(x_1), \ldots, \rho(x_k))$.

[1]Specifically: a partial function ρ from the collection of variables in the language of logical possibility to objects in $\mathscr{M}$, such that the domain of ρ is finite and includes (at least) all free variables in ψ

- $\psi =$ ` $x = y$' and $\rho(x) = \rho(y)$.
- $\psi = \neg\phi$ and ϕ is not true relative to $\mathcal{M}, \rho$.
- $\psi = \phi \wedge \psi$ and both ϕ and ψ are true relative to $\mathcal{M}, \rho$.
- $\psi = \phi \vee \psi$ and either ϕ or ψ are true relative to $\mathcal{M}, \rho$.
- $\psi = \exists x \phi(x)$ and there is an assignment ρ' which extends ρ by assigning a value to an additional variable v not in ϕ and $\phi[x/v]$ is true relative to $\mathcal{M}, \rho'$[2].
- $\psi = \Diamond_{R_1 \ldots R_n}\phi$ and there is another model $\mathcal{M}'$ which assigns the same tuples to the extensions of $\boldsymbol{R}_1 \ldots \boldsymbol{R}_n$ as $\mathcal{M}$ and $\mathcal{M}' \models \phi$.[3]

If we ignored the existence of sentences which demand something coherent but fail to have set models because their truth would require the existence of too many objects[4], we could then characterize logical possibility as follows:

> **Set Theoretic Approximation**: A sentence in $\mathcal{L}$ is true (on some interpretation of the quantifier and atomic relation symbols of the language of logical possibility) iff it is true relative to a set theoretic model whose domain and extensions for atomic relations captures what objects there are and how these atomic relations actually apply (according to this interpretation) and the empty assignment function ρ.

[2] As usual (?) $\phi[x/v]$ substitutes v for x everywhere where v occurs free in ϕ

[3] As usual I am taking $\Box$ to abbreviate $\neg\Diamond\neg$

[4] e.g. categorical descriptions of the intended height of the hierarchy of sets, if one is an actualist about sets and takes the height of the hierarchy of sets to be describable.

Appendix B

Recursive Definition and Isomorphism Lemma

B.1 Recursive Definition Lemma

[We wish to establish the familiar result from set theory that functions (or in our case relations representing functions) can be defined via transfinite recursion, i.e., given a rule for the value of $f(\alpha)$ in terms of $f(\beta)$ for $\beta < \alpha$ there is some function satisfying this rule.] [put in hyperref to well ordering def]

This lemma says that we can use implicitly define how a function behaves on some well ordering [this should be a hyperref to the def of well ordering] $w, \leq$ inductively.

[FIX THIS SO THAT ϕ cant use modal stuff, because I never need that !!!]

Lemma B.1.1. *[this isn't nicely formalized...write it instead like]*

[put a definition somewhere that $\forall x : w(x)$ and $\exists x : w(x)$ are the quantifier restrictions so you don't need to write out the implies each time]

[somewh]

Suppose $w, \leq$ is a well ordering, $\mathscr{L}$ a list of relations such that $w, \leq \in \mathscr{L}$ and $F \notin \mathscr{L}$ and $\varphi(F, x, y)$ is a formula in the language of logical possibility content restricted to $F, \mathscr{L}$ such that $\Box_{\mathscr{L}}(\forall x : w(x)) \left[(\forall x' < x)(\exists!\, z F(x', z))\right] \implies \exists!\, y\varphi(F, x, y)$ then it's logically possible holding fixed the facts about relations in $\mathscr{L}$ that F is a function on w and $(\forall x : w(x))(\varphi(F, x, F(x))$]

[hmm we have to be careful that the unique definition doesn't change in different possible scenarios]

[d] Suppose we have a well ordering $w, \leq$, and an attempted definition by cases $(\forall x)(F'(0) = x \leftrightarrow \phi(x))$ and $(\forall x)(\forall o)(F'(o') = x \leftrightarrow \psi(x)$ where ϕ doesn't mention F, ψ only mentions values of $F'(o')$ for $o' < o$ and the relations F, F' are not in the list $\mathcal{L}$. If it is logically necessary (given the facts about $\mathcal{L}$) that

- *a unique object satisfies $\phi(x)$*
- *$(\forall o)$ if $w(o)$ and o is not the first ordinal and $F(o)$ is defined and functional for all $o < o'$ such that $w(o')$, then there is a unique object which satisfies $\psi(o', x)$*

then it is logically possible (given the facts about $\mathcal{L}$) that F is functional, defined on all of $w, \leq$ and satisfies the above implicit definition. [/d]

[Your proof as written ignores fact that F can't appear in formula in modal comprehension...also doesn't deal with the fact that in different possible scenarios our definition might give different possible values...we use content restriction to ensure this isn't the case]

[I'm just going to sketch proof you fill in...

Using the wrapping trick we define the formula $\phi(x, y)$ to have the same truth conditions as $w(x) \wedge \Diamond_L \left[(\forall x' : x' < x)\varphi(F, x' F(x'))\right] \wedge \varphi(F, x, y)$ and apply Modal Comprehension to ϕ.

now you can use the least element argument u give below...but u need to use the content restriction and uniqueness of F to let you infer that just because $\Diamond_L \varphi(F, x, y)$ holds so too does $\varphi(F, x, y)$...if you run into difficulty doing this let me know

You might want to go into a bit of detail since this is less obvious than one might thing

]

[note that $<$ abbreviates a claim involving $\leq$ and $=$ as usual?]

By Modal Comprehension (and the Wrapping Trick) it is $\Diamond_{\mathcal{L}}$ that $(\forall o)(\forall x)F(o, x)$ iff it is $\Diamond_{\mathcal{L}}$ that F' codes a function which satisfies our stipulation up to o (i.e., $F'(o)$ is the unique object satisfying ϕ and $\forall o'$ if $o' \leq o \wedge \neg o = 0$ then we have $\psi(o', x)$ for some unique x) and $F'(o, x)$.

Now we can show that that $F(o)$ is defined for each ordinal ordered by $\leq$ (and unique so our use of functional notation is justified) essentially as follows (I suppress use of the Wrapping Trick).

Suppose that this is not true. By the fact that $w, \leq$ is a well ordering, there must be a least o at which it fails, so that F does behave like a function on all the objects below o. Then (by Simple Comprehension) we could have F' which behaves like F for as long as F behaves like a function, and takes the first ordinal o at which F does not behave like a function to: the unique object satisfying $\phi(x)$ if o is the first ordinal in $w, \leq$, and the unique object satisfying $\psi(o, x)$ otherwise. [inc: Note that such an object must exist, by the logical necessity claim in the Lemma's assumptions. Thus we know the F' just introduced codes for a function which is defined on o.] This possibility insures that $F(o, x)$, contrary to our initial assumption.

B.2 Isomorphism Lemmas

[again where is hyperref to def of iso]

Lemma B.2.1 (Ismorphism Lemma)**.** *If ϕ is a formula containing only relations $R_1 \dots R_n$ and quantifiers restricted to $R_1 \dots R_n$ outside of all $\Box$s and $\Diamond$s, ϕ' is the result of replacing each R_i in ϕ with a corresponding R'_i (which does not occur anywhere in ϕ), and $x_1, ...x_n$ are all the free variables in ϕ then:*

$$\vdash \langle R_1, \dots R_m \rangle \cong_Z \langle R'_1, \dots R'_m \rangle \rightarrow (\forall x_1)...(\forall x_n)...(\forall x'_1)...(\forall x'_n)[Z(x_1, x'_1) \wedge ...Z(x_n, x'_n) \rightarrow (\phi(x_1...x_n) \leftrightarrow \phi'(x'_1...x'_n)]$$

I will prove this lemma by induction on the complexity of ϕ. Let us say that a formula ϕ is more complex than ψ iff ϕ contains more instances the logical possibility operators $\Box_{...}$ and $\Diamond_{...}$ than ψ or they contain the same number of instances of these operators but ψ is a subformula[1] of ϕ.

I will show that if the lemma holds for all lists of relations $\mathscr{L}$, $R_1...R_n$ and formulas less complex than ϕ, then it holds for ϕ and $R_1...R_n$.

[1] That is, ϕ can be got from ψ by applying the causes for generating a wff for one of the first order connectives

The only significant difficulty we face in this induction is going from [detail about what going from means...maybe more like deriving the truth of the claim for blah given it for blah] ϕ to $\Diamond_L\phi$ statements, as $\Diamond_L\phi$ can be content restricted to a list of relations $R_1...R_n$ (and hence satisfy the assumptions of the lemma) in cases where the sentence ϕ is not so content restricted (so our inductive hypothesis tells us nothing about it directly). To this end we will introduce a modified version of ϕ, $\hat{\phi}$ which will be content restricted to some $\hat{L}$ (with $\hat{L} = \mathcal{L} \cup \mathcal{W} \cup \{U\}$) so that $\Diamond_L\phi$ holds iff $\Diamond_L\hat{\phi}$.

- Specifically, $\hat{\phi} = (\forall x)(x \text{ in } Ext(\mathcal{L}, W_1...W_p) \rightarrow U(x)) \wedge \phi|_U$, where [use the $\in$ symbol in formula..also maybe remind what restriction to U means] $\mathcal{W} = W_1...W_p$ are all the relations occurring in ϕ but not $\mathcal{L}$ and $\phi|_U$ is the result of restricting all the quantifiers in ϕ (outside of $\Diamond$s) to U.
  - For example, if ϕ is $(\exists x)(\exists y)(W(x) \wedge \neg W(y))$, and $\mathcal{L} = L_1, L_2$ then $\hat{\phi} = (\forall x)(\forall y)[L_1(x) \vee L_2(x,y) \vee W(x) \rightarrow U(x) \wedge U(y)] \wedge (\exists x)[U(x) \wedge (\exists y)(U(y) \wedge W(x) \wedge \neg W(y))]$

  Intuitively, $\hat{\phi}$ is a sentence which says that ϕ would be true if the universe were cut back to include only the objects satisfying U.

We will then show that if $\Diamond_L\hat{\phi}$, it's possible to extend any isomorphism Z_0 from L to L' to an isomorphism Z' between $\hat{L} = L \cup W \cup \{U\}$ and $\hat{L}' = L' \cup W' \cup \{U'\}$ (where L' is what you get by replacing each of the R_i in L with the corresponding R_i'). We can then use inductive hypothesis to establish that a version of $\hat{\phi}$ with R_is swapped out for R_i' would be true in this logically possible scenario.

From this, we can derive that $(\Diamond_L \phi)'$ and hence complete a proof that if $\Diamond_L \phi$ then $(\Diamond_L \phi)'$.

Lemma B.2.2 (Z' Building Lemma)**.** *If $\Diamond_{\mathcal{L}} \hat{\phi}$, and Z_0 an isomorphism from $\mathcal{L}$ to $\mathcal{L}'$, then it is possible (holding fixed $\mathcal{L}, Z_0, \mathcal{L}'$) to have $\hat{\phi}$ and Z' extending Z_0 an isomorphism between $\hat{\mathcal{L}} = \mathcal{L} \cup \mathcal{W} \cup \{U\}$ and $\hat{\mathcal{L}}' = \mathcal{L}' \cup \mathcal{W}' \cup \{U'\}$.*

Proof. Assume that Z_0 isomorphically maps $Ext(L)$ to $Ext(L')$ and $\Diamond_L \hat{\phi}$ as above. Let $\mathcal{W} = W_1..W_n$ be all the relations which occur in ϕ but not $\mathcal{L}$. By Ignoring and Pasting, we can have $\hat{\phi}$ while preserving all the facts about $\mathcal{L}, \mathcal{L}', Z_0$ - and hence the fact that Z_0 behaves like an isomorphism.[2]

Enter this $\Diamond_{\mathcal{L},\mathcal{L}',Z}$ scenario. By the Multiple Stipulation Lemma, it is logically possible (holding fixed $\mathcal{L}, \mathcal{L}', Z, U, \mathcal{W}$) to simultaneously have:

- U' applies to those objects which either are in $Ext(\mathcal{L}')$ or satisfy U but don't belong to $Ext(L)$. That is: $(\forall x)[U'(x) \leftrightarrow x$ in $Ext(\mathcal{L}') \vee (U(x) \wedge \neg x$ in $Ext(\mathcal{L}))]$.

- Z' extends Z to a map from U to U', by behaving like the iden-

[2]First we apply Ignoring to go from $\Diamond_{R_{j1} \ldots R_{jm}} \psi|_U$ to $\Diamond_{R_1..R_n, R'_1..R'_n, Z} \psi|_U$ (this is permitted because $\psi|_U$ is content-restricted to a list of relations which does not include any of the relations which we are adding to the subscript of the $\Diamond$). Then we use $\Diamond$ I to infer from the fact that actually $(\langle R_1, \ldots R_m \rangle \cong_Z \langle R'_1, \ldots R'_m \rangle)$ to modal conculsion that $\Diamond_{R1..Rn,R'..R'n,Z}(\langle R_1, \ldots R_m \rangle \cong_Z \langle R'_1, \ldots R'_m \rangle)$.
Finally we are in a position to use the Pasting axiom schema to combine these two scenarios and deduce $\Diamond_{R1,\ldots R_n, R'_1 \ldots R'_n, Z}(\psi * \wedge \langle R_1, \ldots R_m \rangle \cong_Z \langle R'_1, \ldots R'_m \rangle)$ as desired. (Note that the sentence inside our first $\Diamond$ claim, $\psi|_U$, is content-restricted to $R1..Rn, R'..R'n, Z$ plus some some otherwise unused relations $U, \mathcal{W}$ while the sentence inside our second $\Diamond$ claims is completely content-restricted to $R_1..R_n, R'_1..R'_n, Z$, so – as the conditions for the Pasting axiom schema require – there is no 'collision' of terms which are not subscripted in both $\Diamond$s.)

tity function on all the 'newly added' objects which satisfy $\boldsymbol{U}$ but are not in $\boldsymbol{Ext}(\boldsymbol{L})$ (hence are not assigned any value by $\boldsymbol{Z}$). That is $(\forall x)(\forall y)[\boldsymbol{Z}'(x,y) \leftrightarrow \boldsymbol{Z}(x,y) \vee (\boldsymbol{U}(x) \wedge \neg(\exists k)(\boldsymbol{Z}(x,k)) \wedge x = y)]$

$$\boldsymbol{Z}'(x) = \begin{cases} \boldsymbol{Z}(x) & \text{if } (\exists k)(\boldsymbol{Z}(x,k)) \\ \\ x & \text{if } \neg(\exists k)(\boldsymbol{Z}(x,k)) \wedge \boldsymbol{U}(x). \end{cases}$$

- $(\forall \vec{x})(\boldsymbol{W}_1'(\vec{x}) \leftrightarrow (\exists \vec{y})[\boldsymbol{W}_1(\vec{y}) \wedge \boldsymbol{Z}'(x_1, y_1)) \wedge \boldsymbol{Z}'(x_j, y_j))]) \wedge ... \wedge (\forall \vec{x})(\boldsymbol{W}_p'(\vec{x}) \leftrightarrow (\exists \vec{y})[\boldsymbol{W}_p(\vec{y}) \wedge \boldsymbol{Z}'(x_1, y_1)) \wedge \boldsymbol{Z}'(x_j, y_j))])$

  Each predicate $\boldsymbol{W}_i'$ from among $\mathcal{W}' = \boldsymbol{W}_1' ... \boldsymbol{W}_p'$ applies to exactly those objects are the $\boldsymbol{Z}'$-image something which satisfies $\boldsymbol{W}_i$.

It is straightforward to check that this ensures $\langle \boldsymbol{\mathcal{L}}', \mathcal{W}, \boldsymbol{U} \rangle \cong_{\boldsymbol{Z}'} \langle \boldsymbol{\mathcal{L}}', \mathcal{W}', \boldsymbol{U}' \rangle$. And $\hat{\phi}$ must remain true in this new logically possible scenario, since it is content restricted to $\boldsymbol{\mathcal{L}}, \mathcal{W}, \boldsymbol{U}$.

Exiting the two $\Diamond$ contexts introduced above gives us $\Diamond_{\mathcal{L},\mathcal{L}',Z} \Diamond_{\mathcal{L},\mathcal{L}',Z,U,\mathcal{W}}$ ($\boldsymbol{Z}'$ extends $\boldsymbol{Z}_0$ and is an isomorphism between $\hat{\boldsymbol{\mathcal{L}}} = \boldsymbol{\mathcal{L}} \cup \mathcal{W} \cup \{\boldsymbol{U}\}$ and $\hat{\boldsymbol{\mathcal{L}}}' = \boldsymbol{\mathcal{L}}' \cup \mathcal{W}' \cup \{\boldsymbol{U}'\}$). Applying the $\Diamond$ simplification lemma gives us the $\Diamond \boldsymbol{\mathcal{L}}, \boldsymbol{\mathcal{L}}', \boldsymbol{Z}$ claim desured (that $\hat{\phi}$ could be true alongside a suitable $\boldsymbol{Z}'$).

□

Now on to the induction.

The base case, where ϕ is an atomic formula is straightforward.[3]

[3] Suppose ϕ has the form $x_i = x_j$ and $\boldsymbol{R}_1, \dots \boldsymbol{R}_m$ are some atomic relations. Then we need to show $\vdash \langle \boldsymbol{R}_1, \dots \boldsymbol{R}_m \rangle \cong_{\boldsymbol{Z}} \langle \boldsymbol{R}_1', \dots \boldsymbol{R}_m' \rangle \rightarrow (\forall x_i)(\forall x_j)(\forall x_i')(\forall x_j')[\boldsymbol{Z}(x_i, x_i') \wedge \boldsymbol{Z}(x_j, x_j') \rightarrow (x_i = x_j \leftrightarrow x_i' = x_j')]$

But this follows immediately from the definition of isomorphism under $\boldsymbol{Z}$ ($\boldsymbol{Z}$ must be injective).

Suppose ϕ has the form $x_i = x_i$ and $\boldsymbol{R}_1, \dots \boldsymbol{R}_m$ are some atomic relations. Then we need to show $\vdash \langle \boldsymbol{R}_1, \dots \boldsymbol{R}_m \rangle \cong_{\boldsymbol{Z}} \langle \boldsymbol{R}_1', \dots \boldsymbol{R}_m' \rangle \rightarrow (\forall x_i)(\forall x_i')[\boldsymbol{Z}(x_i, x_i') \rightarrow (x_i = x_i \leftrightarrow x_i' = x_i')]$

So is the induction clause for adding the complexity of a formula by applying a sentential connective. Note that if ϕ has the form $\rho \vee \psi$ then ϕ and ψ must contain the same number or fewer $\Box$ and $\Diamond$ operators, so they will be of lesser complexity, so we can assume by induction that Isomorphism Lemma holds for them.[4]

Now we check the induction clause for adding the complexity of a formula by applying a suitably restricted quantifier (the kind which can occur in a formula which satisfies the hypotheses of the Isomorphism Lemma with regard to some relations $R_1, \ldots R_m$).

Specifically, suppose ϕ takes the form $(\forall x)[x \text{ in } Ext(R_1 ... R_n) \rightarrow \psi(x, \vec{y})]$ where $\vec{y}$ are free variables and $R_1, \ldots R_n$ are some relations and the whole formula only contains relations from among $R_1 \ldots R_m$ and quantifiers restricted to $R_1 \ldots R_m$ (occurring outside $\Box$s and $\Diamond$s). Then the component formula ψ also only contains relations from among $R_1 \ldots R_m$ and quantifiers restricted to $R_1 \ldots R_m$ occurring outside $\Box$s and $\Diamond$s). So by inductive hypothesis, we have:

But this follows immediately by first order logic.

Suppose ϕ has the form $R_i(x_1 ... x_n)$ and $R_1, \ldots R_m$ are some atomic relations including R_i. Then we need to show $\vdash \langle R_1, \ldots R_m \rangle \cong_Z \langle R'_1, \ldots R'_m \rangle \rightarrow (\forall x)(\forall y)...(\forall x')(\forall y')...[Z(x, x') \wedge Z(y, y') \ldots \rightarrow (R(x1...xn) \leftrightarrow R'(x1'...xn')$

This also follows immediately from the definition of isomorphic maps (Z must respect the behavior of each R_i).

[4]Suppose ϕ has the form $\neg\psi$ and contains only relations $R_1 \ldots R_n$ and quantifiers restricted to $R_1 \ldots R_n$. Then so does ψ. So, by inductive hypothesis, we already have $\vdash \langle R_1, \ldots R_m \rangle \cong_Z \langle R'_1, \ldots R'_m \rangle \rightarrow [\forall \vec{y} \forall \vec{y}(Z(y_1, y'_1) \wedge ... \rightarrow (\psi(\vec{y}) \leftrightarrow \psi'(\vec{y}')))])$. Then obvious application of first order logic lets us turn this a proof of the above conditional into a proof of $\langle R_1, \ldots R_m \rangle \cong_Z \langle R'_1, \ldots R'_m \rangle \rightarrow [\forall \vec{y} \forall \vec{y}(Z(y_1, y'_1) \wedge \ldots \rightarrow (\neg\psi(\vec{y}) \leftrightarrow \neg\psi'(\vec{y}')))])$ as desired.

Suppose ϕ has the form $\rho \vee \psi$ and contains only relations $R_1 \ldots R_n$ and quantifiers restricted to $R_1 \ldots R_n$. Then so do ϕ and ψ. Thus by inductive hypothesis we have that if $\langle R_1, \ldots R_m \rangle \cong_Z \langle R'_1, \ldots R'_m \rangle$ then $\forall \vec{y} \forall \vec{y}(Z(y_1, y'_1) \wedge ... \rightarrow (\rho(\vec{y}) \leftrightarrow \rho'(\vec{y}'))$ and $\forall \vec{y} \forall \vec{y}(Z(y_1, y'_1) \wedge ... \rightarrow (\psi(\vec{y}) \leftrightarrow \psi'(\vec{y}'))$. Again first order logic lets us derive that $\forall \vec{y} \forall \vec{y}(Z(y_1, y'_1) \wedge ... \rightarrow (\rho(\vec{y}) \vee \psi(\vec{y}) \leftrightarrow \rho'(\vec{y}) \vee \psi'(\vec{y}))$ as desired.

$\vdash \langle R_1, \dots R_m\rangle \cong_Z \langle R_1', \dots R_m'\rangle \rightarrow [\forall x \forall x' \forall \vec{y} \forall \vec{y}'(Z(x,x') \wedge Z(y_1,y_1')... \rightarrow (\psi(x,\vec{y}) \leftrightarrow \psi'(x',\vec{y}'))])$

We want to show that

$\vdash \langle R_1, \dots R_m\rangle \cong_Z \langle R_1', \dots R_m'\rangle \rightarrow ((\forall \vec{y})(\forall \vec{y}')[Z(y_1,y_1') \wedge ... \rightarrow [(\forall x)(x \text{ in } Ext(R_1...R_n) \rightarrow \psi(x,\vec{y})) \leftrightarrow (\forall x')(x' \text{ in } Ext(R_1'...R_n) \rightarrow \psi'(x',\vec{y}'))]$

Assume that Z is an isomorphism between $\langle R_1, \dots R_m\rangle$ and $\langle R_1', \dots R_m'\rangle$ which maps $\vec{y}$ to $\vec{y}'$. Now we prove the needed biconditional by exploiting the fact that every x in $Ext(R_1...R_n)$ is also in $Ext(R_1...R_m)$ and hence in the domain of Z (and similarly every x in $Ext(R_1'...R_n')$ is also in $Ext(R_1'...R_n')$ and hence in the range of Z).

Specifically, we can show that Z maps every counterexample to the LHS of the biconditional to a counterexample to the RHS, as follows. If x in $Ext(R_i)$ and not $\psi(x,\vec{y})$, then for some x' we have $Z(x,x')$. So applying the theorem assumed to exist for inductive hypothesis, we can deduce that not $\psi'(x',\vec{y}')$. And by noting that Z maps any $\vec{v}$ such that $R_i(\vec{v})$ to $\vec{v}'$ such that $R_i'(\vec{v})$ we can deduce that x' in $Ext(R_1'...R_n')$. The same argument works in reverse to show the opposite direction of the biconditional.

Proving the corresponding clause for $\exists$ is now straightforward. Suppose ϕ takes the form $(\exists x)[x \text{ in } Ext(R_1...R_n) \wedge \psi(x,\vec{y})]$ where $\vec{y}$ are free variables. If this formula contains only relations $R_1 \dots R_n$ and quantifiers restricted to $R_1 \dots R_n$ (occurring outside $\Box$s and $\Diamond$s) then so does the corresponding formula $\neg\psi$.

Assume that Z is an isomorphism between $\langle R_1, \dots R_m\rangle$ and $\langle R_1', \dots R_m'\rangle$ which maps $\vec{y}$ to $\vec{y}'$. By the argument regarding $\forall$ above, we can deduce

$(\forall x)(x \text{ in } Ext(R_1...R_n) \rightarrow \neg\psi(x, \vec{y})) \leftrightarrow (\forall x')(x' \text{ in } Ext(R'_1...R'_n) \rightarrow \neg\psi'(x', \vec{y}'))$

So, by first order logic we can infer

$(\exists x)(x \text{ in } Ext(R_i) \wedge \psi(x, \vec{y})) \leftrightarrow (\exists x')(x' \text{ in } Ext(R'_i) \wedge \psi'(x', \vec{y}'))$

Finally, suppose $\phi = \Diamond_{\mathcal{L}}\psi$ and is content restricted to $R_1, \ldots R_m$. We need to prove that:

$\vdash \langle R_1, \ldots R_m\rangle \cong_Z \langle R'_1, \ldots R'_m\rangle \rightarrow (\Diamond_{\mathcal{L}}\psi \leftrightarrow \Diamond_{\mathcal{L}'}\psi')]$

I will assume that $\langle R_1, \ldots R_m\rangle \cong_Z \langle R'_1, \ldots R'_m\rangle$ and then prove the needed biconditional.

First note that existence of this Z isomorphism ensures the possibility (given the facts about $\mathcal{L}, \mathcal{L}', Z$ of a more restricted isomorphism Z_0 between the extensions of our subscripted relations $\mathcal{L}$ and $\mathcal{L}'$ (as needed to apply the Z-Building Lemma above). Because $\phi = \Diamond_{\mathcal{L}}\psi$ and is content restricted to $R_1...R_n$, we know that $\mathcal{L}$ is a sublist of $R_1...R_n$. By modal comprehension, it is logically possible that Z is the restriction of Z_0 to pairs of objects whose first element is in $Ext(\mathcal{L})$ and whose second element is in $Ext(\mathcal{L}')$. By first order logic we can check that this Z behaves like an isomorphism between $Ext(\mathcal{L})$ and $Ext(\mathcal{L}')$. Now note that the biconditional $(\Diamond_{\mathcal{L}}\psi \leftrightarrow \Diamond_{\mathcal{L}'}\psi')$ is content restricted to $\mathcal{L}, \mathcal{L}'$, so to prove it is true in our original situation it suffices to prove that it is true in this $\Diamond_{\mathcal{L},\mathcal{L}',Z}$ context.

Enter the above $\Diamond$ context. I will now argue for the $\rightarrow$ direction of the needed biconditional (the other direction can be proved in exactly the same way) via the following big picture strategy. I will suppose that $\Diamond_{\mathcal{L}}\psi$ and then derive that it is logically possible for our 'portable' version $\hat{\psi}$ to be true. Applying the Z'-Building Lemma then tells us that we can have $\hat{\psi}$

remain true while Z' behaves like an isomorphism between $\hat{\mathcal{L}} = \mathcal{L} \cup \mathcal{W} \cup \{U\}$ and $\hat{\mathcal{L}}' = \mathcal{L}' \cup \mathcal{W}' \cup \{U'\}$. Then I will appeal to our inductive hypotheses to this content restricted sentence derive that a corresponding sentence (call it $\hat{\psi}'$) is true in this logically possible scenario. Finally, I will argue from this to the conclusion that $\Diamond_{\mathcal{L}'}\psi'$.

Suppose that $\Diamond_{\mathcal{L}}\psi$. To show that $\Diamond_{\mathcal{L}}\hat{\psi}$, enter the above $\Diamond_{\mathcal{L}}\psi$ scenario. Let $\mathcal{W}$ be a list of all the relations in ψ but not $\mathcal{L}$. By Simple Comprehension we have $\Diamond_{\mathcal{L},\mathcal{W}}(\psi \wedge (\forall x)(U(x) \leftrightarrow x = x))$. Entering this further $\Diamond$ context, we can derive that $\hat{\psi}$ [by simple first order logic]. Dropping out of both $\Diamond$ contexts successively gives us $\Diamond_{\mathcal{L}}(\Diamond_{\mathcal{L},,\mathcal{W}}\hat{\psi})$. Applying $\Diamond$ simplification to this gives us $\Diamond_{\mathcal{L}}\hat{\psi}$.

Now we have $\Diamond_{\mathcal{L}}\hat{\psi}$ and a suitable isomorphism Z_0, so we can deploy the Z-Extending Lemma to get that it is logically possible (holding fixed the facts about $\mathcal{L}, \mathcal{L}', Z$) that $\hat{\psi}$ and Z' extends Z_0 and functions like an isomorphism between $\hat{\mathcal{L}} = \mathcal{L} \cup \mathcal{W} \cup \{U\}$ and $\hat{\mathcal{L}}' = \mathcal{L}' \cup \mathcal{W}' \cup \{U'\}$. Now $\hat{\psi}$ is content restricted to $\mathcal{L} \cup \mathcal{W} \cup \{U\}$, so by inductive hypothesis we have:

$\vdash \langle \mathcal{L}, \mathcal{W}, U \rangle \cong_Z \langle \mathcal{L}', \mathcal{W}', U' \rangle \rightarrow (\hat{\psi} \leftrightarrow \hat{\psi}')$

where $\hat{\psi}'$ replaces $\mathcal{L}, \mathcal{W}, U$ with $\mathcal{L}', \mathcal{W}', U'$ Accordingly, we can infer from $\hat{\psi}$ to $(\hat{\psi})'$. Thus we have $\Diamond_{\mathcal{L}'}\hat{\psi}'$ within our previous scenario.

Finally, it remains to go from $\Diamond_{\mathcal{L}'}(\hat{\psi})'$ to $\Diamond_{\mathcal{L}'}\psi'$. Note that $(\hat{\psi})'$ was formed by taking the version of ψ which restricts all quantifiers (outside of $\Box$es and $\Diamond$s) to U and then asserts U applies to all objects, *and then* replacing each relation in $\mathcal{L}, \mathcal{W}, U$ with the corresponding primed relation. Thus it differs from ψ' in two ways: it asserts that U' to be the whole universe and it replaces all the relations W_i in $\mathcal{W}$ (i.e., all relations which occur in ψ but only inside some $\Box$ or $\Diamond$ context) with corresponding W_i'.

First we enter the $\Diamond_{\mathcal{L}'}(\hat{\psi})'$ context and then use the Cutback axiom schema to derive that $\Diamond_{\mathcal{L}',\mathcal{W}',U'}(\forall x)U'(x)$. This inference is permitted because we can derive that U' applies to all objects in $Ext(\mathcal{L}', W', U')$ from $\hat{\psi}'$ by simple first order logic[5]. Also, within this $\Diamond$ context, we still have $\hat{\psi}'$, because this sentence is content-restricted to $\mathcal{L}', W', U'$. Thus we can infer that $\psi'[\mathcal{W}/\mathcal{W}']$ holds in this scenario by simple first order logic [as above]. Thus (stepping back) we know that $\Diamond_{\mathcal{L}'}\Diamond_{\mathcal{L}',\mathcal{W}',U'}\psi'[\mathcal{W}/\mathcal{W}']$. By relabling and $\Diamond$ dropping we can infer that $\Diamond_{\mathcal{L}'}\psi'$, as desired. This completes our proof of $\Diamond_{\mathcal{L}}\psi \to \Diamond_{\mathcal{L}'}\psi'$.

Swapping the roles of $\mathcal{L}$ and $\mathcal{L}'$ lets us deduce the other direction of the needed bi-conditional in exactly the same way. So, finally, discharging our assumptions for $\to$I gives us $\vdash \langle R_1, \dots R_m\rangle \cong_Z \langle R_1', \dots R_m'\rangle \to (\Diamond_{\mathcal{L}}\psi \leftrightarrow \Diamond_{\mathcal{L}'}\psi')$, as desired.

[5] Technically, we also need to check that at least one thing satisfies U'. But we know this because $(\exists x)(x = x)$ is a theorem (so at least one thing exists) and $\hat{\psi}$ contains the claim that everything that exists satisfies U'.

Appendix C

Helpful Facts about Hierarchies

C.1 On Isomorphisms Between Initial Segments

def R isomorphically maps from an initial segment of a well ordering $\boldsymbol{ord}_i, \leq_i$ to an initial segment of another well ordering $\boldsymbol{ord}_j, \leq_j$

- $\boldsymbol{ord}_i'$ applies in a way that is closed down.
- $\leq_i'$ behaves just like $<_i$, but only applies to objects satisfying $\boldsymbol{ord}_i'$.

def R isomorphically maps from an initial segment of V_i to an initial segment of V_j iff $\Diamond_{V_i,V_j}(V_i' \leq_{set} V_i \wedge V_j \leq_{set} V_j' \wedge \langle V_i' \rangle \cong_R \langle V_j' \rangle)$

Lemma C.1.1 (Initial Segment Lemma). *If $\mathscr{V}(V)$ and*

- *all $\boldsymbol{ord}_k$ are $\boldsymbol{ord}$s and all $\boldsymbol{set}_k$ are sets*
- *$\in_k, \leq_k, @_k$ are just the restrictions of $\in, \leq, @$ to the $\boldsymbol{ord}_k$ and $\boldsymbol{set}_k$*

- *V_k is closed downwards under $\leq$ in terms of which ords it contains.* $(\forall o)(\forall o')[ord_k(o) \land o' \leq o \to ord_k(o')]$

- *V_k contains exactly the sets corresponding to these ord_k* $(\forall x)[set_k(x) \leftrightarrow (\exists o)(ord_k(o) \land @(x, o))]$

then $V_k \leq_{set} V$

Proof. This fact can be shown fairly straightforwardly by checking through definitions. Note that if $set_k(b)$ and $a \in b$ then b is available at some $ord_k(o)$ and a is available at some $o' \leq o$, so we have $ord_k(o')$ as well.

[should I say more about this? e.g. also note that we can show that V_k satisfies thickness by using Ign to note that any possible application for a predicate P could apply while holding fixed the facts about V, and then invoking the fact that V satisfies thickness.] □

C.2 Isomorphism Agreement Lemmas

Lemma C.2.1 (Hierarchy Agreement Lemma)**.** *Suppose that R and R' are both isomorphic maps from an initial segment of V_0 to an initial segment of V_1. Then* $\forall x \forall y \forall y' R(x, y) \land R'(x, y') \to y = y'$

Suppose that R and R' are both isomorphic maps from an initial segment of V_0 to an initial segment of V_1. We will (in essence) prove the Lemma above by induction on the ord_0 in V_0, showing that if R and R' agree on all ords below x, they must agree on x.

More specifically, suppose for contradiction that R and R' are differing isomorphisms as above. Then R and R' disagree with regard to a set (or ordinal) that is available at some ordinal level o. By simple comprehension, it is logically possible (holding fixed all the relations in V_0 R and R') that a property B (for Bad) applies to each ordinal o in V_0 iff there is some set at level o, or ordinal $o' \leq o$ which R and R' disagree on. Thus there must be a least Bad o, by the fact that the ord_0 are well ordered[1]. R and R' will disagree on some set_0 or ord_0 which available at level o, while agreeing on every set_0 and ord_0 available at some level $o' < o$.

Suppose x is an ord. Then by the fact that R and R' agree everywhere below o, $x = o$. Since R and R' agree on all $o* < x$ (and map to and from initial segments of V_0, V_1), both must map x to the least ord_1 above every ord_1 they map some $o* < x$ to, so they both must map x to the same thing. [inc: more explanation?]

Suppose x is a set. By the fact that R and R' agree on all sets available below o, we can derive that R and R' must agree on x, using only first order logic.

Suppose, for contradiction, that $R(x) = y$ and $R'(x) = y'$, where $y \neq y'$. Then there is some z in y but not y' (or vice versa). Assume the former w.l.o.g. Because R maps between initial segments, R maps some z^{-1} to this z. Because R is an isomorphism and $z \in_1 y$, we have $z^{-1} \in_0 x$ and hence z^{-1} is available at some level below the level at which x first appears. Because R' maps between initial segments it maps this z^{-1} to something.

Now we have a dilemma. If $R'(z^{-1}) \in y'$ then R and R' must already disagree about z, since we chose z to not be in y'. This contradicts our choice of x as something that first occurs on the lowest level at which R and

[1] Inc?[more explanation for how this is derivable?]

$\boldsymbol{R}'$ disagree. On the other hand, suppose that $\neg\boldsymbol{R}'(z^{-1}) \in y' = \boldsymbol{R}'(x)$. This contradicts the fact that $\boldsymbol{R}'$ is an isomorphism because we have $z^{-1} \in x$.

Thus the logical possibility of contradiction follows from our assumption, so we can conclude that our assumption was false: $\boldsymbol{R}$ and $\boldsymbol{R}'$ never disagree.

Lemma C.2.2 (Well Ordering Agreement Lemma)**.** *Suppose that R and R' are both isomorphic maps from an initial segment of a well ordering* $\boldsymbol{ord}_0, <_0$ *to an initial segment of* $\boldsymbol{ord}_1, <_1$. *Then* $\forall x \forall y \forall y'\, \boldsymbol{R}(x, y) \wedge \boldsymbol{R}'(x, y') \rightarrow y = y'$

We can prove that isomorphisms between initial segments of *well orderings* must also agree, by exactly the same argument, simply replacing all mentions of isomorphisms between hierarchies V_0 and V_1, with mentions of isomorphisms between well orderings $\boldsymbol{ord}_0, <_0$ and $\boldsymbol{ord}_1, <_1$, and then dropping the clause about what happens if x is a set above.

C.3 V Comprability Lemma

Lemma C.3.1 (V Comprability Lemma)**.** *If* $\mathcal{V}(V) \wedge \mathcal{V}(V')$ *then* $\Diamond_{V,V'}$ *(R isomorphically maps from an initial segment of* V *to* V' *or vice versa)*

By modal comprehension we can (in effect) have $\boldsymbol{R}(x, y)$ iff it is possible to isomorphically map an initial segment of V_0 to one of V_1 in a way that takes x to y.

This $\boldsymbol{R}$ will be our isomorphism Z between initial segments of V_0 and V_1. First we check that it has all the necessary properties to be an isomorphism, as follows.

We can show that R is functional and one to one, by using the Isomorphism Agreement Lemma above (which says that isomorphisms between initial segments of a hierarchies V_0 and V_1 such that $\mathscr{V}(V_0)$ and $\mathscr{V}(V_1)$ must agree with one another in places where they are both defined), as follows. [I give this argument using variables, but one can cash it out using the Wrapping Trick in the familiar way.]

Suppose for contradiction that we had some x such that $R(x, y)$ and $R(x, y')$ where $\neg y = y'$. Then (by the Pasting Lemma) it would be possible to have two different isomorphisms R'_0 [and R'_1] from an initial segments of V_0 to an initial segment of V_1 which disagree with each other on x.[2][This contradicts (a relevant instance of) the Isomorphism Agreement Lemma] Similarly, if we had $R(x, y)$ and $R(x', y)$ where $\neg x = x'$, then it would be possible to have two different isomorphisms R'^{-1} from initial segments of V_1 to initial segments of V_0 which disagreed on x.

Next, we can show that R respects $\in$ and other relations. Suppose, for contradiction, R did not respect $\in$. Then, by simple choice, we could have P apply to some unique pair of objects x and y which are a counterexample. [alt: such that $(\exists x')(\exists y')[R(x, x') \wedge R(y, y') \wedge \neg(x \in_0 y \leftrightarrow x' \in_1 y')]$.] One of the pair x and y is $\geq$ the other (by trichotomy on $<_0$). Suppose w.l.o.g. it's y. Consider an R' witnessing the fact that R is defined on y. Because

[2] More specifically, we can cash out the above argument without quantifying in as follows. By simplified choice, we could have Q applies to a single x on which R is not uniquely defined. And we can have a two-place relation P applies only to (x,y) for some $R(x, y)$, and then P' applies only to some (x, y') for some y' such that $R(x, y')$ and $\neg y = y'$. By our characterization of R, we have $\Diamond_{V_0,V_1,R,P} R'$isomaps initial segments and for x and y st $P(x, y)R'(x, y)$ and $\Diamond_{V_0,V_1,R,P'} R''$ isomaps initial segments and for x and y' s.t. $P'(x, y')R''(x, y)$. By ignoring (to add P'/P to make the subscripts the same) and then Pasting we can have R and R' behaving as above simultaneously. But this violates the isomorphism agreement lemma. Contradiction.

this is an isomorphism (and its domain is closed down), it takes x and y to a pair of objects x' and y' such that $x \in_0 y \leftrightarrow x' \in_1 y'$. But then so does $\boldsymbol{R}$. And by the fact that $\boldsymbol{R}$ is 1-1, these are the only x' and y' which $\boldsymbol{R}$ takes x and y to. Thus we have $(\forall x')(\forall y')[R(x, x') \wedge R(y, y') \rightarrow (x \in_0 y \leftrightarrow x' \in_1 y')]$. This contradicts our choice of x and y.

We can also show that $\boldsymbol{R}$ maps from an initial segment, by using the Initial Segment Lemma above. The first two requirements for the lemma are clearly satisfied.

[note abuse of notation between $\boldsymbol{R}$ as a function and a relation?]

To show that the range of $\boldsymbol{R}$ is closed downward under $\leq$, suppose there was an $\boldsymbol{ord}_0$ $\boldsymbol{o}$ such that $R(o)$ is defined and for some $o' \leq_1 o$, $R(o')$ is not defined. By simplified choice, we could have $\boldsymbol{Q}$ apply to a unique such pair o and o'. By the fact that $\boldsymbol{R}$ is defined on o, there could be an $\boldsymbol{R}'$ which maps from an initial segment including $\boldsymbol{o}$. But then because this $\boldsymbol{R}'$ maps from an initial segment including o'. So, by our characterization we have $\boldsymbol{R}$ also defined on $\boldsymbol{o}$. Contradiction. [note that we are again using a version of the wrapping trick?]

Exactly the same reasoning (where we consider a witnessing $\boldsymbol{R}'$ and note that it is defined on an initial segment) shows us the final condition is satisfied. $\boldsymbol{R}$ is defined on a set_1 x iff there is some $\boldsymbol{o}$ such that $\boldsymbol{R}$ is defined on $\boldsymbol{o}$ and $@_1(x, o)$.

By a symmetrical argument, $\boldsymbol{R}$ maps to an initial segment. Thus $\boldsymbol{R}$ maps between initial segments.

Now it only remains to verify that $\boldsymbol{R}$ either maps from all of V_0 or to all of V_1. Suppose not, for contradiction. That is: there is something in V_0

which R doesn't map from *and* something in V_1 which R doesn't map to.

Then [(by the fact that the ord_0 are well-ordered by $\leq_1$)] there is a least ordinal level, o, such that R fails to be defined for some set or ordinal available at or below o. By using modal comprehension and the simplified choice lemma we can show that there is such a level, and then enter a logically possible scenario on which the predicate P labels a *unique* object x occurring at this level o, for which $R(x)$ is not defined).[3]

Now I will argue that it is logically possible for R' to isomorphically map the initial segment of V_0 up through this ord_1 o (where the unique object satisfying $P(\cdot)$ first becomes available) to an initial segment of V_1. This yields the desired contradiction, because (by our characterization of R) it implies that R *is* defined on the unique object satisfying $P(\cdot)$, and this contradicts our characterization of P.

[explain "available" above]

By simple comprehension, it is logically possible to have R' such that $R'(x, y)$ iff x is a set_1 or ord_1 available at o (where o is the least ord_1 that R doesn't map from as above) and y is:

- $R(x)$, if x is a set_1 or ord_1 available below o.

- the least ord_1 which R doesn't map to, if x is o.

  Note that such an ord_1 must exist, for the following reason. By hypothesis, there is some x which R doesn't map to. If this x is not already

[3]By modal comprehension we can (in effect) say that an ord o is bad iff $R(x)$ is not defined for some set or ord x available at o. Then (because R doesn't map from everything) there are some bad ords. By the facts that the ords in V_0 are well ordered, there is a least bad o. By Simple Choice, P can apply to a unique object x at this least bad level o, such that $R(x)$ is not defined.

an ord_1, it must be available some ord_1 o_x which R also doesn't map to. So, by the well ordering constraint built into $\mathscr{V}(V_1)$, there is must be least ord_1 which R doesn't map to.

- the set in V_1 which contains exactly the R-image of the elements of x, if x is a set_1 first available at o in V_0.

  Note that all the elements of x *have* R-images because they occur at or below o, the first level of V_0 where R fails to be defined.

  Also note that a set in V_1 containing exactly these R-images must exist (and, indeed, be available at the ordinal level $R'(o)$ specified above). We have just shown that there's a least ordinal level ($R'(o)$) which R does not map to. Since R maps onto an initial segments of V_1 (as proved above), it only maps onto set_1s (and ord_1s) which are available at some ordinal $<_1$ $R'(o)$. Thus, by the thickness of V_1, a set containing exactly the R-images of the elements of x exists (and, indeed, is available at $R'(o)$).

Finally, we can check that this possible R' isomorphically maps V_0 up through o [alt: the set_0, ord_0 available up at o (the first ordinal level on which R is not defined)] to an initial segment of V_1 (and hence yields the desired conclusion that R is defined on the unique object satisfying P):

- R' is functional. This immediate from the definition above.

- R' maps *from* an initial segment of V_1. By the Initial Segment Lemma, it suffices to show the following:

- The domain of $\boldsymbol{R}'$ contains all ordinals below ordinals it contains [alt: is closed downwards, in terms of ordinals]. This is immediate from the fact that $\boldsymbol{R}$ maps from initial segments and our choice of $\boldsymbol{o}$ as the first $\boldsymbol{ord}_0$ level at which R is not defined.
- The domain of $\boldsymbol{R}'$ contains *all* sets available at ordinals it contains. Consider an arbitrary $\boldsymbol{set}_1$ x, which is available at some $\boldsymbol{o}'$ in the domain of $\boldsymbol{R}$. Then $\boldsymbol{o}' \leq \boldsymbol{o}$, by the fact that $\boldsymbol{R}$ is only maps from $\boldsymbol{set}_1$s and $\boldsymbol{ord}_1$s available at $\boldsymbol{o}$. This implies that x must also available at $\boldsymbol{o}$, by the conditions on the relationship between $\boldsymbol{set}_1$s and $\boldsymbol{ord}_1$s built into $\mathscr{V}(V_1)$[4].
- The domain of $\boldsymbol{R}'$ contains *only* $\boldsymbol{set}_1$s and $\boldsymbol{ord}_1$s available at some $\boldsymbol{ord}_1$ it contains. This is immediate from the fact that the domain of $\boldsymbol{R}'$ includes $\boldsymbol{o}$ and only includes $\boldsymbol{set}_1$s and $\boldsymbol{ord}_1$s available at $\boldsymbol{o}$]

- $\boldsymbol{R}'$ is 1-1. This follows straightfowardly from the fact that $\boldsymbol{R}$ is 1-1 and our definition of $\boldsymbol{R}'$ (using the fact that the $\boldsymbol{sets}_1$ are extensional and R respects $\in$). [say more?]

- $\boldsymbol{R}'$ is respects $\in, @, \leq$. This follows straightforwardly from the fact R does and our definition of $\boldsymbol{R}'$.

- $\boldsymbol{R}'$ maps onto an initial segment of V_1. By the Initial Segment Lemma it suffices to show the following:

  - When $\boldsymbol{R}'$ maps to an ordinal then it maps to all ords $\leq$ it.

[4]More specifically, the thinness condition insures us that all the elements of x occur below o', hence they occur below o. Thus, we can show that there is a set_1 which contains them at level o, by the thickness condition [using the Wrapping trick to avoid quantification over o' inside modal contexts, in the usual way].

First note that everything in the range of R is also in the range of R'. Whenever we have $R(x, y)$, x must occur below o (and hence be available at o), since it is possible to have a mapping between initial segments of V_0, V_1 which includes x but this mapping cannot include o (by our choice of o).[5] Thus we will also have $R'(x, y)$ in this case.

Now consider an arbitrary situation where $ord_1(x)$ and $y \leq_1 x$ and x is in the range of R'. By our characterization of R', there are only two possibilities for x. If x is $R'(o)$ then it follows our choice of $R'(o)$ as the first ordinal level which R doesn't map onto all of, that y is in the range of R. So y is in the range R'. If x is in the range of R, then y is in the range of R, by fact that R maps onto an initial segments proved above. So, again, y is in the range of R'.

– R' maps to *all* sets available at that ordinals it maps to. Consider an arbitrary ord x in the range of R' and an arbitrary y available at x. If x is also in the range of R then the needed result follows from the fact that R maps to an initial segments. If not, then x is $R'(o)$, i.e., the first ordinal above the initial segment which R maps to. Since y is available at $R'(o)$ then its elements occur at least one level lower, so R maps some $z_1, z_2 ...$ in V_1 to these

[5]More rigorously, suppose for contradiction, that x is not available at o. By the fact that $\mathscr{V}(V_1)$ we know that x first occurs at some an ordinal level o'. And we can infer that $o' > o$. But (by our characterization of R) it is possible to have a mapping between initial segments of V_0, V_1 which includes x. Then it must also include o'. And since $o' > o$, it must also include o. Thus we can have a mapping between initial segments which includes o, so R is defined on o, contradicting our specification of o. [I have suppressed an obvious use of IGN to derive $\Diamond$ claims which allow importation of the fact that $o' > o$].

elements of y. These z_1, z_2... in V_1 occur below o, by our choice of o as the fact that $\boldsymbol{R}$ maps from initial segments. So by the thickness condition on V_0, there is a set_0 which collects z_1, z_2... available at level $\boldsymbol{o}$. $\boldsymbol{R'}$ maps this set_1 to y, because its $@_1 o$ and $\boldsymbol{R}$ maps the individual z_1, z_2... to the elements of y.

- If $\boldsymbol{R'}$ maps to *only* sets which are available at ordinals it maps to. Let x be an arbitrary set in the range of $\boldsymbol{R'}$. If x is also in the range of $\boldsymbol{R}$, the fact to be shown is immediate from the fact that $\boldsymbol{R}$ maps to an initial segment. If x is in the range of $\boldsymbol{R'}$ but not $\boldsymbol{R}$, then (by specification of $\boldsymbol{R'}$) its elements get mapped to by $\boldsymbol{R}$, so they are all in the range of $\boldsymbol{R}$. Because $\boldsymbol{R}$ maps to an initial segment, this means that all these elements occur below the first ord which $\boldsymbol{R}$ doesn't map onto all of. And by our note on the definition of $\boldsymbol{R'}$ above this ordinal is $\boldsymbol{R'}(o)$. Thus x is available at an ordinal in the range of $\boldsymbol{R'}$.

Thus it is (indeed) logically possible for $\boldsymbol{R'}$ to isomorphically map the initial segment of V_0 up through $\boldsymbol{ord}_1$ $\boldsymbol{o}$ (where the unique object satisfying $\boldsymbol{P}(\cdot)$ first becomes available) to an initial segment of V_1. This yields the a contradiction, as desired, because (by our characterization of $\boldsymbol{R}$) it implies that $\boldsymbol{R}$ *is* defined on the unique object satisfying $\boldsymbol{P}(\cdot)$.

C.4 One More Layer Lemma

Lemma C.4.1. *If V behaves like a hierarchy of sets, it is possible to form a V' such that $V' \geq_{set} V$ by adding one more layer of classes.*

Proof. Our strategy here will be to invoke the possibility of a layer of classes over the elements satisfying set. We will then take the set' to include all the objects satisfying set together with those of these classes which can't be identified with existing sets, with membership defined in the obvious fashion. We will extend the ords by adding a single new object which is an ord' but not an ord.

More formally, we use the Layer of Classes axiom schema to establish the logical possibility (holding fixed the facts about V) that $class$ and $\in_c$ apply as if to layer of classes over the sets.

Then we use simple comprehension[6] to establish the logical possibility (holding fixed the facts about V as well as $class$ and $\in_c$) of relations V' satisfying the following conditions:

- $(\forall x)(set'(x) \leftrightarrow set(x) \vee class(x) \wedge (\forall y)[set(y) \rightarrow (\exists z)\neg(z \in y \leftrightarrow z \in_c x)]$
- $(\forall x)(\forall y)[x \in' y \leftrightarrow set'(x) \wedge set'(y) \wedge (x \in y \vee x \in_c y)])$
- $(\forall x)(ord'(x) \leftrightarrow ord(x) \vee class(x) \wedge (\forall y)(\neg y \in_c x))$
- $(\forall x)(\forall y)(x \leq' y \leftrightarrow ord(x)' \wedge ord'(y) \wedge (x < y \vee ord(x) \wedge \neg ord(y)))$
- $(\forall x)(\forall y)(x @' y \leftrightarrow set(x)' \wedge ord'(y) \wedge (x @ y \vee set(x) \wedge \neg ord(y)))$

Now it is tedious but fairly straightforward to verify that $\mathscr{V}(V')$ and indeed $V' \geq_{set} V$. [I will describe the two places where modal/non FOL reasoning is used]

For example, to check that that the ord's under $\leq'$ satisfy the least element condition for being a well ordering, note that by ignoring if we had

[6]as per the Multiple Definitions Lemma

$\Diamond_{ord',\leq'}$(there is no $\leq$-least happy object), then one would have $\Diamond_{ord',\leq',ord,\leq}$(there is no <-least happy object). Now we can argue as follows: if some of the set's are happy either only our new ord' which is not an ord is the only one (in which case it would be a $\leq'$-least happy element) or some ords would be happy (in which case there would be a $\leq$-least happy element, by the fact that the ord are well ordered by $\leq$, and this would be a $\leq'$-least happy element as well.

To check that the set's satisfy Fatness. Suppose (while holding fixed the facts about $V, class, \in_c, V'$) that the predicate P applies to objects available at levels below our o. First consider the case where o is our newly introduced ordinal o. Then there is a class which contains exactly those objects satisfying P. Either this class is a set' and contains in the sense of $\in'$ exactly those objects which it contains in the sense of $\in_c$, so there is some $(\exists k)(set'(k) \wedge (\forall k')(k' \in' k \leftrightarrow happy(k'))$, or this class is co-extensive with some set, and since every set is also a set' the condition above must still obtain.

On the other hand, suppose that o is not our new ordinal (in the sense of $\in$'). Then the fact that $\mathscr{V}(V)$ ensures that there is a set k which contains (in the sense of $\in$) all the happy sets. And this same k is a set' which contains exactly the happy set's in the sense of $\in'$. For the happy $sets'$ are exactly the happy sets (since all the happy set's are available at a level below o where o is strictly below our new ordinal) and in the special case where y is a set we have $(\forall x)(x \in y \leftrightarrow x \in' y)$, so k contains them all in the sense of $\in'$.

Putting those two arguments together, we have $\Box_{V,class,\in_c,V'}$(happy applies only to set' below some ord' o → there is a set' at' o which contains exactly the happy objects). Now this sentence is content restricted to $V', happy$, so by IGN we can infer that the corresponding $\Box_{V'}$ claim holds true, as desired.

[check whether I use happy in def. or something else?]

C.5 Hierarchy Extending Lemma

If $\mathscr{V}(V_a) \wedge \mathscr{V}(V_b)$ then it's possible (holding fixed V_a, V_b) that there is a V_{a+} extending V_a and Z' functioning like an isomorphism from V_b to an initial segment of V_{a+}

By the V Comparison Lemma we either have V_b isomorphic to an initial segment of V_a already (so the desired conclusion is insured by considering the trivial extension of V), or we have V_a is isomorphic to an initial segment of V_b. So we can assume that

$\Diamond_{V_a,V_b}$ Z isomorphically maps V_a to an initial segment of V_b.

In the special case where V_a and V_b are disjoint, we can prove the desired result as follows. Consider a possible extension V_{a+} of V_a which adds all the 'extra' objects in V_b which are not in the image of V_a under Z. The relation Z' could then map from all of V_{a+} to all of V_b by acting like Z where Z is defined, and like the identity map on all the new objects we have added. And we can fix the application of relations like $\in_{a+}$ in the obvious disjunctive way (e.g., $(\forall x)(\forall y)[x \in_{a+} y \leftrightarrow x \in_a y \vee x \in_b y \vee (\exists w)(Z(x,w) \wedge w \in_b y)]$) and check that all the desired results follow.

By simple comprehension, it is $\Diamond_{V_a,V_b,Z}$ that:

- $(\forall x)(set_{a+}(x) \leftrightarrow set_a(x) \vee set_b(x) \wedge (\forall y)[set_a(y) \rightarrow \neg Z(y,x)]$
- $(\forall x)(ord_{a+}(x) \leftrightarrow ord_a(x) \vee ord_b(x) \wedge (\forall y)[ord_a(y) \rightarrow \neg Z(y,x)]$
- $(\forall x)(\forall y)[Z'(x,y) \leftrightarrow Z(x,y) \vee x = y \wedge (set_{a+}(x) \wedge set_b(x) \vee ord_{a+}(x) \wedge ord_b(x))]$
- $(\forall x)(\forall y)[x \in_{a+} y \leftrightarrow \exists x' \exists y' Z'(x,x') \wedge Z'(y,y') \wedge x' \in_b y']$
- $(\forall x)(\forall y)x <_{a+} y \leftrightarrow \exists x' \exists y' Z'(x,x') \wedge Z'(y,y') \wedge x' <_b y'$
- $(\forall x)(\forall y)x @_{a+} y \leftrightarrow \exists x' \exists y' Z'(x,x') \wedge Z'(y,y') \wedge x' @_b y'$

The characterization of Z' above makes it easy - if tedious - to check that Z' behaves like a 1-1 onto function from $Ext(V_b)$ to $Ext(V_{a+})$.

One final wrinkle remains. If V_a and V_b overlap, it may be that some of the objects which play the role of high-level sets within V_b (i.e., sets which are not in the image of Z) are also set_as. In this case we cannot add them to V_a as new objects. Our intended isomorphism Z' from V_{a+1} to V_b will not be functional, because it will associate each such set_a with two different objects: itself and Z(itself).

We can fix this problem by making one small modification. First we using the Layer of Classes to establish the logical possibility of adding a layer of classes over the set_a and set_b (where none of these classes are identical to any object satisfying set_a or set_b). Then we specify V_{a+} by adding to V_a the *singleton class* of each set_b which is not in the image of Z, rather than adding these set_bs themselves. This guarantees that our V_{a+} will contain a genuinely new object corresponding to each set_b which is not in the image of Z. We can then define the behavior of these singleton classes in terms of the behavior of the single set_b which belongs to them, just as above. □

C.6 Mass Hierarchy-Combining Lemma

Lemma C.6.1. *For any indexed plurality of V_is indexed by the objects satisfying some relation I , it would be logically possible to have a larger V_Σ, such that a)V_Σ is a V and b) each V_i is isomorphically mappable to some portion of V_Σ*

More specifically, suppose that we have a collection of hierarchies indexed by the objects satisfying some relation I, as generated by the Hierarchy Combining Lemma.

$\Box_{set',\in'...I}(\forall x)(I(x) \wedge (\forall \vec{v})(set(\vec{v}) \leftrightarrow set'(\vec{v}, x)) \wedge (ord(\vec{v}) \leftrightarrow ord'(\vec{v}, x)) \wedge ...) \rightarrow \mathscr{V}(set, \in, @, ord, <))$

Then $\Diamond_{set',\in'...I}[\mathscr{V}(V_\sigma) \wedge$ E isomorphically maps each of the V_i associated with $set', \in' ...I$ to an initial segment of V_Σ.

Proof. Our strategy will be to generate V_Σ by taking *equivalence classes* which group together all the different set_i which play analogous roles in different V_i. We will consider equivalence classes under the a relation $\sim$, which relates pairs of analogous sets. Specifically, let $x \sim y$ iff x is a set or ordinal in some V_i, y is a set or ordinal in some V_j, and it is possible to isomorphically map the objects at or below x's level V_i to the objects at or below y's level V_j while taking x to y.

By modal comprehension, a two place relation $\sim$ could apply so that

$x \sim y$ holds iff it is logically possible (holding fixed the facts about all our inded $V*$s) for there to be an isomorphism from an initial segment of V_i that includes to an initial segment of V_j that includes y.

At this point we need to introduce objects representing the equivalence classes. In set theory, the equivalence classes themselves would automatically be sets and thus qualify, but here we must instead introduce a layer of classes over the objects within the various V_i [Should I spell this out in terms of ext and seti?]. This layer of classes contains a class containing just the elements of each equivalence relation and we let the relation EQ map each set x or ord o belonging to some V_i to the unique class which contains it, and is closed under $\sim$. It is easy to verify that $\sim$ is an equivalence relation.[7]

[Here you should just give the definition of V_Σ in terms of representatives of the equivalence classes.] [Do you still think so?]

Our giant hierarchy V_Σ will be formed by grouping together sets (and ordinals) in different V_is which 'play an analogous role' in the sense defined by $\sim$, and taking these to be the elements of V_Σ. We will then allow relationships ($\in_i, <_i$ and $@_i$) between original sets and ordinals to induce facts abot how analogous relations ($\in_\Sigma, <_\Sigma$ and $@_\Sigma$) relate resulting equivalence classes. Specifically, my argument begins by using multiple applications of simple comprehension (combined as per the Multiple Definitions Lemma) to show that it is logically possible (while holding fixed all the facts about the V_i) that:

- add a layer of classes above all the V_i

[7]everything $\sim$ itself (consider the identity map), if $a \sim b$ and b$\sim$c then a$\sim$c (consider composing the isomorphic mappings witnessing the truth of $a \sim b$ and b$\sim$c) and if $a \sim b$ then $b \sim a$ (consider the inverse of the isomorphic mapping)

- let the relation Eq(x,a) pair up each object x with its equivalence class under $\sim$ (i.e., $Eq(x,a) \leftrightarrow class(a) \wedge (\forall y)(y \in a \leftrightarrow x \sim y)$).

- say that something is a set_Σ iff it the Eq of some set_i.

- say that something is an ord_Σ iff it the Eq of some ord_i.

- say that $a \in_\Sigma b$ iff a and b are both set_Σs and something in a is a member of something in b within some V_i(i.e., if $\exists x \exists y Eq(x,a) \wedge Eq(y,b) \wedge x \in_i y$) for some i in the index).

- say that $a @_\Sigma b$ iff a is a set_Σ, b is an ord_Σ and something in a occurs at something in b within some V_i,

- say that $a <_\Sigma b$ iff a and b are both ord_Σs and something in a is less than something in b within some V_i

In what follows, I will sometimes write $[x_i] = a$ [try to avoid the new notation...just restrict your attention to equivalence classes and you should be able to manage] [looking at how short stuff is below, do you still think I should avoid this?] instead of $Eq(x_i, a)$, and use claims about $[x_i]$ to abbreviate corresponding claims about the unique object satisfying $Eq(x_i, \cdot)$ (i.e., the unique class containing exactly those k such that $x_i \sim k$).

Now we need to show that if V_Σ satisfies the conditions above, then has all the further properties required by the Lemma:

A. Eq isomorphically maps each V_i to some initial segment of V_Σ.

It is straightforward to check that Eq provides an isomorphic mapping from each V_i to an initial segment of some initial segment of V_Σ.

First we should note that Eq is functional and maps from all of V_i: Clearly for each set_i or ord_i, x there is a corresponding class of y such that $x \sim y$. Furthermore, if $Eq(x, a)$ and $Eq(x, b)$ then $a = b$, by the our specification of Eq and the fact that our layer of classes is extensional (both a and b must contain exactly those y s.t. $x \sim y$). This justifies my notational assumption that for each set_i (or ord_i) x_i, there is exactly one set_Σ (or ord_Σ) $[x_i]$

We can also note that Eq maps V_i onto an initial segment of V_Σ. Suppose that $[y_i] = y$ and $x \in_\Sigma y$. Then we can show that Eq maps something in V_i to x as follows. By the definiton of $\in_\Sigma$, there is some is some j s.t. $x_j \in_j y_j$ where $y_i \sim y_j$. But this possible map between initial segments must take x_j to something, so we have $x_j \sim x_i$ for some x_i in V_i. Thus there is some x_i in V_i which Eq maps to x. [The same argument works for $x \leq y$ and $@(x, y_)$]

Eq is one-to-one when restricted to V_i. Suppose that $[x] = [y]$ (i.e., $\exists a Eq(x, a) \wedge Eq(y, a)$) for some x and y both in V_i. Then it is possible to have an isomorphic mapping between initial segments of V_i which takes x to y. By pasting, this isomorphism could exist alongside the identity isomorphism of V_i. By the Isomorphism agreement lemma it must agree. Thus we have $x = y$.

Eq respects $\in$ when restricted to V_i. Let x and y be arbitrary set_is. One direction is easy; if $x \in_i y$, then $[x] \in_\Sigma [y]$, immediately, by definition of $\in_\Sigma$.

To get the other direction, suppose that $set_i(x_i) \wedge set_i(y_i) \wedge [x_i] \in_\Sigma [y_i]$. Then there's some V_j containing $x_j \sim x_i$ and $y_j \sim y_i$ such that $x_j \in_j y_j$.

Consider possible isomorphisms of initial segments F witnessing $x_i \sim x_j$

and G witnessing $y_i \sim y_j$. By pasting we could have them side by side. Composing these isomorphisms of initial segments gives us one that maps x_i to something $F(x_i) \in_j y_j$, and then to something $G^{-1}(F(x_i)) \in_i y_i$. It is easy to check that this result of composing isomorphisms between initial segments is itself an isomorphism between initial segments.

As above, know that it must agree with the identity isomorphism of initial segments on V_i (by pasting and the isomorphism agreement lemma), so we have $x_i = G^{-1}(F(x_i))$ and hence $x_i \in_i y_i$, as desired. [The same argument works for $x \leq y$ and $@(x, y_)$]

B. $\mathscr{V}(V_\Sigma)$

First we must check that the ord_Σ are well ordered. We can prove most of these by exploiting the fact that the ord_i in each V_i are well ordered, and the fact that for each x in V_Σ we have $x = [x_i]$ for some x_i in a V_i, such that $\mathscr{V}(V_i)$ and Eq isomorphically maps V_i to an initial segment of V_Σ.

For example, we can check that the ord_Σ satisfy the least element property as follows. Suppose P applies to some ord_Σ o. Then there must be a $<_\Sigma$-least ord_Σ which P does not apply to, for the following reason. Then, by our construction of V_Σ we have $o = [o_i]$ for some x_i in a V_i. We know the ordinals in V_i are well ordered. So, if we consider the property of belonging to an equivalence class that satisfies P, we know there must be a $\leq_i$ least ord_i with this property. Call it m_i. Clearly $[m_i]$ satisfies P, and we can derive that it is the least ord_Σ satisfying P. For if there were any ord_Σ below $[m_i]$ satisfying P, then Eq would have to pair it with an ord_i below m_i whose equivalence class satisfies P, and this contradicts our choice of m_i.

The other properties needed for well ordering are checkable in a similar way, except for trichotomy, which we prove as follows:

$a <_\Sigma b$ or $b <_\Sigma a$ or a=b. $a = [a_1]$ in some V_i, $b = [b_j]$ in some V_j. By the V-Comparison Lemma, one can one isomorphically map all the ordinals in V_i up to a_i onto some segment of the ordinals in V_j up to b_j (or vice versa) by some mapping f. In the former case we have have either $f(x) <_j y$ or $f(x) = y$, so we have $a <_\Sigma b$ or $a = b$. In the latter case, we have either $f(y) <_j x$ or $f(y) = x$, so we have $a <_\Sigma b$ or $a = b$. Thus trichotomy holds.

Now it remains to show that the set_Σ satisfy the Fatness and Thinness conditions. We can do this by using the fact that Fatness and Thinness hold within each V_i, and every object in V_Σ lies within an initial segment of V_Σ which is isomorphic to some V_i.

> Fatness Requirement: $\Box_V(\forall o)[(\forall x)P(x) \rightarrow (\exists o')(ord(o') \wedge o' < o \wedge @(x, o'))] \rightarrow [(\exists y)@(y, o) \wedge (\forall z)(P(z) \leftrightarrow z \in y)]$

Suppose only set_Σs below some ord_Σ o are happy. We need to show there is a set_Σ at o which contains exactly these sets.

Consider some V_i containing an o_i such that $ord(o_i) \wedge [o_i] = o$. Because this V_i satisfies Fatness, it contains a $set_i(h_i)$ containing exactly those x_i such that $set_i(x_i) \wedge happy([x_i])$ and x_i occurs below o_i.

Now we can show that $[h_i]$ must contain all and only the happy set_Σs, by using the fact that Eq isomorphically maps V_i to an initial segment of V_Σ including o.

Let a be an arbitrary happy set_Σ. By hypothesis, we have $a@_\Sigma o'$ for some $o <_\Sigma o$, so a is in the image of V_i, hence $a = [a_i]$. By our characterization of

h_i, we have $a_i \in_i h_i$ and hence $a = [a_i] \in_\Sigma [h_i]$.

Conversely, suppose that $a \in_\Sigma [h_i]$. Then a is in the image of V_i, because Eq maps on to initial segments of V_Σ. So we have $a = [a_i]$ and $a_i \in_i h_i$. Then by our characterization of h_i we have $happy([a_i])$ and hence $happy(a)$.

Thinness Requirement:

- $(\forall x)[set(x) \rightarrow (\exists o)@(x, o)]$.

  Suppose that $set_\Sigma(x)$. Then Eq isomorphically mps some V_i onto an initial segment of V_Σ such that we have some $[x_i] = x$. Because V_i satisfies thinnes we have an o_i such that $x_i @_i o_i$. Thus we have $@_\Sigma([x_i], [o_i])$.

- $(\forall x)(\forall o)(\forall z)[@(x, o) \wedge z \in x \rightarrow (\exists o')(o' < o \wedge @(z, o'))]$.

  Suppose $@_\Sigma(x, o)$ and $z \in_\Sigma x$. Then Eq isomorphically maps some V_i onto V_Σ such that we have some $[o_i] = o$. Because Eq maps onto an initial segment we can infer that there are also have some $[x_i] = o$ and $[z_i] = z$. Because Eq is an isomorphism we have $@(x_i, o_i) \wedge z_i \in_i x_i$. Because V_i satisfies thinness we have o_i' such that $o_i' <_i o_i$ and $@_i(z_i, o_i')$. But because Eq is an isomorphism this implies that $[o_i'] <_\Sigma o \wedge @_\Sigma(z, [o_i'])$ as desired. Thus there is a $o' = [o_i']$ with the properties desired.

- $(\forall x)(\forall y)(set(x) \wedge set(y) \rightarrow x = y \vee (\exists z)[set(x) \wedge \neg(z \in x \leftrightarrow z \in y))$

  Consider arbitrary set_Σs x and y. By the first point above, x occurs at some ord_Σ o, and y at o'. By the fact that the ord_Σ obey trichotomy (proved above) $o < o'$ or $o = o'$ or $o > o'$. Assume w.l.o.g that $o < o' \vee o = o'$. Eq isomorphically maps some V_i containing o_i such that $[o_i'] = o'$ onto an initial segment of V_Σ. By our hypothesis it also associates this means that it also associates objects in $V_i{:}o$ with an ord_i and x and y with set_i. By the fact that the set_i satisfy extensionality we have a z_i such that $\neg(z_i \in_i x_i \leftrightarrow z_i \in_i y_i)$. And because Eq is an isomorphism this implies that there is a $z = [z_i]$ such that $\neg([z_i] \in_\Sigma x \leftrightarrow [z_i] \in_\Sigma y)$

□

C.7 Fleshing out Lemma

Lemma C.7.1. *Suppose that $\leq$ well-orders the objects satisfying W. It is logically possible (fixing the facts about $W, \leq$) to have $\mathscr{V}(V')$ such that the $ord', \leq'$ are isomorphic to the $W, \leq$.*

Proof. I will prove this claim by a kind of inductive argument. Using modal comprehension, we can say (something equivalent to) an object o satisfying P is Good iff it is logically possible to build a V whose spine is to isomorphic to the $\langle P_*, \leq_* \rangle$ structure of objects in the well-ordering up to and including o. Before using the Wrapping Trick to handle quantifying in, we could express this idea as follows:

$$(\forall x)[W(x) \to (Good(x) \leftrightarrow \Diamond_{W,\leq}[\mathscr{V}(V') \wedge (\exists x)(\forall y)(\forall z)[(W_*(y) \leftrightarrow y \leq x) \wedge (y \leq_* z \leftrightarrow y \leq z \wedge z \leq x \wedge y \leq x)\langle ord', \leq' \rangle \cong_Z \langle W_*, \leq_* \rangle])]$$

I will then argue that that all objects in the well ordering are good. Suppose not, for contradiction. By the fact that we are dealing with a well ordering, there is a least o which is not Good[8].

[8][peter says remove?] By simple comprehension, Bad could apply to all the objects

The $\leq$-least o which is not good cannot be the least object in $W, \leq$, because it is trivially possible to build a V whose spine is the first object in the well ordering (and whose sets are just the empty set).[9]

Thus we can assume that there are some ordinals below the least o which is not good. By the fact that everything below o is Good, for each $o' < o$ it is possible to have a hierarchy $V_{o'}$ satisfying whose spine is the series of objects $o'' \leq o'$ satisfying P.

Thus, we can apply the Combinatorial Replacement axiom to get that it's possible to have simultanious witnesses to all these logically possible hierarchies. Then we can use the V-Combining Lemma above, to get the logical possibility of a big V_Σ which a relation Eq isomorphically maps all these smaller V_is onto initial segments of.

Finally, we can apply the one more layer of classes lemma to this to generate the needed V' from $V\sigma$. Now it only remains to show that the $ord', \leq'$ in this newly hierarchy are isomorphic to[10] the objects $o' \leq o$ satisfying W.

First we will show that the ord_Σ in the V_Σ can be isomorphically mapped to the objects satisfying W which are strictly $\leq$ below o. It is easy to check that an initial segment of a well ordering is still a well ordering. Thus we can apply the Well-Ordering Comprability Lemma (proved in C.3) to get that at least one of these structures is isomorphic to an initial segment of the other.

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which are not Good. By the definition of well ordering there is a least Bad object. This is a least object that is not Good. The claim that there is a least object which is not Good is content-restricted to Bad.

[9] This follows by using layer of classes to get an object distinct from the first ord o to the empty set, taking o to be the sole ord' and defining $\in'$ and $@'$ in the obvious way.

[10] or rather, can be isomorphically mapped to

- Suppose that all the W below o map to an initial segment of the $ord_\Sigma, \leq_\Sigma$, and some $o*$ s.t. $ord_\Sigma(o*)$ doesn't get mapped to. Then $o* = [o_i]$ for o_i in a V_i some $i \leq o$. But then there is an isomorphism of initial segments from the $W \leq$ this i to the ord_i to the ord_Σ up to $o*$. Combining this isomorphism of initial segments from $W, \leq$ to $ord_\Sigma, \leq_\Sigma$ with the inverse of the previous one gives us a isomorphism from an initial segment of $ord_\Sigma, \leq_\Sigma$ which does not include o* to one which does. This disagrees with the identity isomorphism on $ord_\Sigma, \leq_\Sigma$, which contradicts the Isomorphism Agreement Lemma (as applied to well orderings).

- Similarly, suppose that all the $ord_\Sigma, \leq_\Sigma$ map onto an initial segment of the W strictly below o, and some such $o*$ doesn't get mapped to. Then there is a V_{o*}, and an isomorphism of initial segments which takes $o*$ to the top ord_o* in it. Composing with Eq gives us an isomorphism of initial segments which takes the ords up through $o*$ to the $ord_\Sigma, \leq_\Sigma$. But now composing this with the inverse of our original isomorphism gives an isomorphism of initial segments which takes the some o' strictly below $o*$ to the o' up to and including $o*$. This disagrees with the identity isomorphism on $W, \leq$, and thus contradicts the Isomorphism Agreement Lemma (as applied to well orderings)

Thus the ord_Σ in the V_Σ can be isomorphically mapped to the objects satisfying W which are strictly $\leq$ below o. We can extend this isomorphism between V_Σ and the $o' < o$ to an isomorphism from the ord_o in $V' \geq_{set} V_\Sigma$ to the $o' \leq o$, by mapping o to the single extra ordinal in V_o but not V_Σ, and checking that the result is an isomorphism. $\square$

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