HAMKINS' ANALOGY BETWEEN SET THEORY AND GEOMETRY: PLURALISM BY LEVELING UP?

SHARON BERRY

Abstract. In [10] set theorist Joel David Hamkins uses considerations about forcing arguments, together with an analogy between set theory and geometry to motivate his set-theoretic multiverse program. I'll argue that Hamkins develops the latter (familiar) analogy in an unusual way, that promises to motivate multiverse theory in particular (rather than merely some form of pluralism). He suggests what I'll call a leveling up approach to mathematical pluralism which motivates distinctive features of his multiverse theory. However, I'll question whether Hamkins' multiverse proposal ultimately delivers leveling up. Then I'll sketch a counterpossible counterfactual-based twist on Hamkins multiverse which does provide relevant leveling up.

1. Introduction

In work like [10], set theorist Joel David Hamkins develops an influential multiverse approach to set theory. On this view, there are many different hierarchies of sets, and there's no fact of the matter about whether certain set-theoretic statements are true, beyond the fact that they are true of some hierarchies of sets within the multiverse and false in others. On this view (contrary to more traditional forms of set theoretic Platonism) there's no full-width intended hierarchy of sets, which contains all possible subsets of sets it contains —or even all subsets its copy of the natural numbers. Rather, for *every* set-theoretic universe V in the multiverse, there's a forcing extension — a 'fatter' universe of sets that includes all sets in V but also a certain missing subset of some infinite set already included in V. Hamkins gives two main motivations for this view in [10]: an appeal to the phenomenology of forcing arguments and an analogy between set theory and geometry.

In §2 of this paper, I'll review some basic facts about Hamkins multiverse project and then note that Hamkins develops the long- familiar pluralist analogy between set theory and geometry in an elaborate and somewhat unusual way. Specifically, he seems to advocate what I'll call a 'leveling up' (rather than 'leveling down') pluralism about both geometry and set theory. This position is both interesting in its own right and seemingly argumentatively useful to Hamkins. For, it promises to let him motivate choosing multiverse theory in particular (which appears to provide a kind of leveling up) over other forms of set theoretic pluralism – not just adopting some form of set theoretic pluralism.

In §3 I propose a way of filling in Hamkins' suggestive sketch of *geometrical* pluralism via leveling up. In §4, I'll question whether Hamkins' multiverse theory actually provides leveling up pluralism about set theory. And in §4 I'll sketch a counterpossible-counterfactual variant on Hamkins' multiverse theory which faces various challenges, but does clearly qualify as leveling up pluralism about set theory.

I don't claim the latter view is correct (I personally favor a far more truth-value realist position [3]). But I'll suggest that it has some appeal and illuminatingly contrasts with Hamkins' multiverse theory in a few ways.

2. Background

2.1. **Hamkins' Platonist Multiverse.** Let us begin with some background about Hamkins' proposal.

Hamkins' multiverse view is, as he says, a form of Platonism[10]. It combines ontological realism (the view that mathematical objects like sets exist) with significant truth value anti-realism (the view that that there isn't a single right answer to many questions in the language of set theory)[1]. Hamkins' approach differs from both formalism and conventional Platonist approaches to set theory, in taking there

[1]For example, Hamkins writes, "In this article, I shall argue for... the multiverse view, which holds that there are diverse distinct concepts of set, each instantiated in a corresponding set-theoretic universe, which exhibit diverse set-theoretic truths. Each such universe exists independently in the same Platonic sense that proponents of the universe view regard their universe to exist."[10]

to be many differently structured universes of sets, that are all on equal footing[2] – rather than a single intended set-theoretic universe.

Most centrally, Hamkins claims that every set-theoretic universe V has a forcing extension $V[G]$ which adds some [3]) 'missing subsets' to a set in V. This contrasts with more conventional (single universe) forms of Platonist set theory. Such conventional single universe Platonists take the intended hierarchy of sets to already contain all possible subsets of sets it contains, so it cannot be thus expanded. Accordingly, they regard forcing arguments as telling us about how *countable models* of the ZFC axioms (within the true universe of sets) could be extended – or about boolean valued models[4] – not how the total universe we are currently working in could be extended.

In addition to this claim about forcing extensions, Hamkins advocates some even bolder closure conditions on the multiverse in [10] like the following. Every universe looks countable from the point of view of some larger universe (Countability Principle). And every universe's copy of the ordinals looks ill-founded from the point of view of some larger universe (Well-Foundedness Mirage).

Hamkins motivates his claim that all universes have forcing extensions (and this indirectly his multiverse proposal) in three ways in [10]. First, he appeals to the phenomenology of making forcing arguments. He reports that his experience of making such arguments suggests contact with a genuine universe of sets, properly extending whatever universe he is working in, when making these arguments [5]. Second, he argues that standard ways of reporting theorems proved by forcing arguments have changed, so as to be more susceptible to a multiversist reading.

[2]That is, all these variant universes are on equal metaphysical footing, the mathematical study of them counts as equally set theoretic etc. This is not to say that all universes in the multiverse are equally interesting, beautiful, or useful to study.

[3]More specifically, Hamkins accepts the "Forcing Extension Principle" that "For any universe V and any forcing notion P in V, there is a forcing extension $V[G]$, where $G \subseteq P$ is V-generic."[10]

[4]Here I mean models of set theory that don't assign determinate binary facts about elementhood.

[5]By 'working in', I mean quantifying over and using 'set' and 'element' to mean set and element within.

But third (and most importantly for this paper) Hamkins appeals to a claimed "very strong analogy"[10] between historical changes in attitudes towards geometry and the revision in understandings of set theory which he is advocating.

2.2. **Hamkins' Analogy.** So let's now turn to that analogy. Using a comparison with geometry to motivate and explain set theoretic pluralism/truthvalue anti-realism is very common. Set theoretic pluralists often say things like, 'there's no general right answer to set-theoretic questions like the continuum hypothesis, just like there's no general fact about whether the parallel postulate is true'[6]. However, in this section, I'll note that Hamkins fleshes this familiar comparison out in a somewhat eccentric way, which is both interesting in its own right and potentially useful to his argument.

Hamkins writes that there's, "a very strong analogy between the multiverse view in set theory and the most commonly held views about the nature of geometry"[10]. He supports this claim by describing three stages of progress in attitudes to geometry which (he proposes) we should emulate in the case of set theory, as follows.

In the first stage, mathematicians take there to be a unique background geometrical universe, which fixes the intended interpretation of 'point' and 'line' in all contexts[7]

In the second stage, mathematicians embrace a limited geometrical pluralism. They accept that studying variants on Euclid's axioms can have mathematical interest and be part of mathematics. However, they regard these variant axiom systems as having a secondary status, merely being true on certain "toy models"[10]/unintended interpretations, like ones which interpret 'line' as meaning great circle on the surface of a sphere in Euclidean space. And they deny theorems proved in these axiom systems have *genuine geometrical content*. Instead, they regard such theorems as a legitimate part of mathematics, but merely showing

[6]See, for example, [6]

[7]Hamkins writes, "For two thousand years, mathematicians studied geometry, proving theorems about and making constructions in what seemed to be the unique background geometrical universe."[10]

something about provability and models (as work in any syntactically consistent axiom system would).

Finally, in the third stage, mathematicians accept full geometrical pluralism, which regards work in various geometrical axiom systems as equally legitimate and fully "geometrically meaningful" [10]. Hamkins describes this change as arising from a process where, "geometers gained experience in the alternative geometries, developing intuitions about what it is like to live in them" [10].

I think this picture of the transition from traditional to pluralist geometry is interesting in two ways.

First, according to this picture fully adopting geometrical pluralism requires coming to see theorems in variant axiom systems as having *genuine geometrical content* (as not just providing mathematical knowledge of some kind). Accordingly, we face the question: what does it mean to regard a theorem as having genuinely geometrical (or set-theoretic) character? In a way this question is naturally motivated by disputes between multiverse theory and traditional single universe Platonist set theory (which Hamkins compares to the disagreement between stage two to stage three attitudes to geometry). For note that the traditional single-universe Platonist will heartily agree that studying variant set-theoretic systems is a legitimate part of *mathematics*, and indeed reveals truths about hierarchies of sets (extending countable models of ZFC). So their real disagreement with the multiverse theorists seems to concern whether such mathematical work has a further feature – which we might call expressing genuinely *set theoretic* content.

Second, Hamkins description of a path to geometrical pluralism suggests what I'll call a leveling up (as opposed to leveling down) understanding of geometrical pluralism, in the following sense.

According to what I'll call *leveling down pluralism* about geometry, we can go from traditional views to a suitable form of geometrical pluralism by (so to speak) mere subtraction. We come to accept variant axiom systems as 'fully geometrical'. But doing this is purely a matter rejecting/debunking traditional claims that attributed previously privileged axioms systems a (geometrically relevant) special status. On this view, going pluralist doesn't require any substantive change to our stage two (pre-pluralist) understanding of work in *non-Euclidean axiom systems*[8]. It just requires renouncing (or re-classifying as irrelevant to mathematics) traditional claims about ways Euclidean geometry was uniquely correct or intended (e.g., rejecting assumptions that Euclid's axioms expressed a priori metaphysically necessary truths on the intended physical interpretations of 'point' and 'line'). We cease to regard alternative geometries as telling us about *mere* toy models, by renouncing the distinction between intended and toy models.

In contrast, Hamkins' description of a three stage path to geometrical pluralism suggests a different species of geometrical pluralism, which I'll call pluralism by leveling up. Becoming a geometrical pluralist in this sense will still involve some debunking and rejection of assumptions about a traditionally favored axiom systems (at the very least, we renounce anti-pluralist claims that these systems are special). But it also requires somehow enriching or changing your attitude to traditionally disfavored geometrical axiom systems, by coming to see work in these variant systems as telling you about something different from mere provability and toy models. For note that Hamkins describes a transition to full geometrical pluralism arising from a process of "geometers gain[ing] experience in the alternative geometries, developing intuitions about what it is like to live in them"[10]. And he seemingly continues to accept and appeal to an important distinction between mere toy models and something else ('full grown' models) after going pluralist[9].

In the next section, I'll suggest a way of fleshing out this (admittedly somewhat cryptic) vision of leveling up geometrical pluralism. I think this view is interesting

[8]Obviously, it requires the Cambridge-change like transition of ceasing to take these axioms to exist alongside other axioms with a favored status like being true on the uniquely intended interpretation.

[9]For example, in the case of set theory (which is clearly supposed to be parallel to that of geometry) he characterizes his multiverse theory as taking variant axioms to be true of intended as 'full grown' models of set theory, unlike traditional approaches which merely take these theories to have toy models.

and worth exploring (even aside from any connection to Hamkins) as a counterpoint to more common leveling down pluralism about geometry. However, I don't claim leveling up pluralism is overall preferable. Prima facie, I think both styles have some appeal.

For example, leveling-down pluralism has a kind of logical positivist adjacent Gordian knot cutting appeal – promising to help naturally eliminate many prima facie vexing philosophical questions. For example, if we entirely reject the distinction between intended and unintended interpretations of 'point' and 'line', then we don't need an account of the meaning of these bits of geometrical vocabulary. Also, we can (but need not) give a simple picture of the relationship between physics and geometry, by saying the following. Scientific theories are free to appeal to mathematical facts by incorporating bridge laws. But no particular principles connecting geometrical terms to physical facts or empirical expectations have special a priori or quasi-analytic status.

On the other hand, this very simplicity of leveling down pluralism will strike some as more implausibly Procrustian than attractively Gordian knot cutting in a few ways. For example (by rejecting all a priori preferred physical interpretations of geometrical vocabulary), the simple view just sketched threatens to make the meaningfulness of claims like 'space turned out not to be Euclidean' (in contexts where no particular physical interpretation of 'point' and 'line' has been made salient) a mystery[10]. Also, it's not clear how advocates of leveling-down pluralism can understand work in variant axiom systems as having genuine geometrical character.[11]

[10]Note that the intuitive meaning of such claims about the geometrical structure of space seems to go beyond bland claims that there are *some* physical interpretations of 'point' and 'line' on which Euclid's axioms fail. Rather it seems to involve (something like) an idea that Euclid's axioms fail *on all the most natural/intended* physical interpretations 'point' and 'line'.

[11]They could say that all consistent axioms in the language of geometry (trivially) count as having genuine geometrical character. But this doesn't fit with use of geometry to pick out a somewhat interesting and useful natural kind within mathematics.

Alternately (taking inspiration from algebra) they could say any axiom system which proves certain core sentences (any interpretation which makes these sentences come out true) in the language of geometry will have genuine geometrical character. However, they won't be able to use traditionally expected connections to physical space to characterize (or motivate) the boundaries of geometry – as wikipedia still does by saying "Geometry is a branch of mathematics concerned with properties of space such as the distance, shape, size, and relative position of figures."[14]. For

3. Geometrical pluralism by leveling up

In this section, I will try to provide a concrete example of leveling up geometrical pluralism. My story takes inspiration from Hamkins' remarks (as I will indicate below), but extrapolates beyond them in ways I don't claim he would endorse.

Let's start with the question what does it mean to see work in variant axiom systems as having genuine geometrical character/being geometrically meaningful?[12]

As we have seen, Hamkins intriguingly associates starting to accept variant geometries as "geometrically meaningful" with getting a sense of "what it is like to live [in]" them[10]. Similarly, in the case of set theory (which he claims is closely analogous), he describes forcing arguments as motivating set theoretic pluralism by giving one an experience of glancing into "new set-theoretic worlds" and studying "what it would be like to live in them"[10]. Admittedly sometimes this talk of *living in* a different geometry/set theory does just seem to be a picturesque way of speaking about *working in* (i.e., proving things in) variant systems. However, this is not always the case. For example, he claims that set theorists can "reason about a forcing extension by jumping into it and reasoning **as though** they were living in that extension" (emphasis mine). Thus, he seems to allow for a three-way contrast between

- implicitly representing a universe where $\neg$CH, by reasoning about a suitable forcing relation on the sets in your current background universe, which you take to satisfy ZFC and CH

they dismiss traditional connections between geometry and physical reality (like default favored physical interpretations of 'distance' and 'shape') as irrelevant to pure mathematics. C.f. Einsten on the appeal of connecting geometry to ideas about physical measuring (rather than treating all interpretations of geometrical vocabulary on par) in [7]

[12]One might think the obvious answer to this question is that an axiom system is geometrically meaningful iff there is some acceptable precisification of geometrical vocabulary on which all its principles come out true. But, even if correct, this leaves us with an obvious remaining question about which interpretations are acceptable and why.

- 'jumping in' to this forcing extension by reasoning as if you lived in a universe where CH by working with suitable axioms (e.g. proving things from ZFC $+\neg$CH)
- **actually living** in a set universe where $\neg$CH

Accordingly, a leveling up pluralist might identify coming to see variant axiom systems as fully geometrical with developing the ability to see proofs from these axioms as telling us about what it would be like to 'actually live in' a world with correspondingly different geometry.

Ok so far, but what could such talk of 'actually living in' a different geometry (or imagining doing so) mean? I think considering what people mean when they say things like, "Scientists originally assumed we lived in a Euclidean world, but they were wrong" (in contexts where no specific physical interpretation of 'point' and 'line' has been mentioned) suggests a possible answer to this question. Perhaps "actually living" in a geometrically different universe, means living in a universe where facts about the structure of space are different, so that different geometrical axioms could come out true *under their traditionally intended physical interpretations* (i.e., when 'point' and 'line' are given a priori/default favored physical interpretations).

In more detail, we might say that someone can positively level up their attitudes to some variant geometrical axiom systems (come to regard them as fully geometrical) via the following process.

First, do some debunking which clears the ground for geometrical pluralism as follows. Admit that there are a range of different a priori live options for what the structural constraints on physical space (studied by traditional geometry) might be. Decide, on this basis, that we should separate traditional geometry into a priori and empirical portions. Empirical study is needed to determine the kinds of facts traditional geometry hoped to discover a priori (laws constraining the physically necessary/possible configuration of physical objects, points and lines etc.). So if we want to retain the idea that mathematics is an a priori activity, it seems natural to propose that geometry (insofar as it's a branch of mathematics) should equally include the investigation of all a priori live options for these laws of physical geometry. This opens conceptual space for work in various incompatible axiom systems to qualify as genuinely geometrical.

Second, build up a suitable positive relationship to some variant geometrical axiom systems. Begin to see the hypothesis that these variant axiom systems truly describe physical geometry (in the sense of being true under the traditionally intended interpretations of 'point', 'line' etc.) as (loosely) coherent and imaginable – perhaps something that creatures with our limited faculties of a priori insight can't and shouldn't rule out a priori. This change in attitude to an axiom system involves a substantive psychological shift which might (but need not) involve coming to use mental pictures in a new way.

Putting all this together yields a form of leveling up pluralism which differs from the simple (leveling down) geometrical pluralism sketched above as follows:

- Traditional expectations about the intended physical applications of geometry remain relevant to mathematics (though not in a way that picks out a unique object for geometers to study).
- Work in various axiom systems can be regarded as having "genuine geometrical character" iff it can be regarded as telling us about an (in some sense) a priori conceivable live option for the structure of physical space (specifically, telling us about what else would be true if the structure of physical space made relevant axioms true while preserving traditionally expected connections between physical reality and geometry). Thus:
  - There's (potentially) a motivated natural kind contrast between axiom systems whose theorems can vs. can't be seen as having geometrical character[13].

[13]However, there may be some blurriness about this boundary and a spectrum of more vs. less physically natural physical interpretations of point and line, leading to vagueness about where this boundary lies.

– A substantive psychological shift is involved in switching from thinking about a geometrical axiom system as merely having toy models (like the stage two pre-pluralist geometer in Hamkins' story) to regarding work in this axiom system as having genuinely geometrical character. One must acquire (something like) the ability to imagine physical space having different structures, and come to see work in variant geometrical axiom systems as informative about imaginable possibilities where physical space makes variant axioms true (under all sufficiently traditionally intended physical interpretations of terms like 'point' and 'line).

So I think this way of fleshing out Hamkins' remarks about leveling up to geometrical pluralism fits the text reasonably well. It also avoids the pair of worries for simple leveling down pluralism mentioned above (by giving a motivated account of 'genuine geometrical character' and the default meaning of claims that physical space satisfies certain geometrical axioms).

I also think this way of fleshing out geometrical pluralism via leveling up fits decently well with Hamkins' suggestion that working in different geometries can give mathematicians a sense of, "what it's like to live in [different geometries]."[10] and thereby prompt pluralism. For it seems fairly plausible that accruing instinctive skill, speed and familiarity with seeing consequences in a certain geometrical axiom system might significantly help enable the above gestalt switch to finding it imaginable that physical space satisfies these axioms (on natural/intended physical interpretations of terms like 'point' and 'line').

Now let's return to our main topic: set theory and the multiverse. In addition to being interesting in its own right, developing (and suggesting we mirror) a *leveling up* approach to geometrical pluralism promises to do argumentative work for

Hamkins. In the next section, I'll review why multiverse theory initially seems better positioned than other forms of set-theoretic pluralism (like formalism) to mirror leveling up pluralism about geometry, and then question this appearance.

4. Is Hamkins' Multiverse Set Theoretic Pluralism Via Leveling Up?

Suppose you like leveling up pluralism about geometry, and want to mirror it in the case of set theory. Can we do so? Does Hamkins' multiverse theory actually provide a form of leveled-up pluralism about set theory?

Hamkins' multiverse view can initially seem to provide a kind of 'leveling up' route to set-theoretic pluralism, analogous to leveling up pluralism about geometry. For acknowledging that there are universes corresponding to different set axioms sounds like granting them some kind of new positive status. And Hamkins lists taking forcing arguments to put one in contact with 'fully grown' models of set theory – rather than merely countable toy models – as an advantage of the multiverse view over traditional single universe approaches to set theory (which he compares to stage two pre-pluralist attitudes to geometry)[14]

However, Hamkins doesn't say much to explicate this claim to (distinctively) regard forcing arguments as telling us about full grown set-theoretic universes. And cashing it out turns out to be somewhat problematic. Note that Hamkins cannot simply say that 'full grown' models are supposed to be ones which satisfy the traditional iterative hierarchy conception of the width of the set theoretic universe (by containing, at each level, sets corresponding to all possible ways of choosing from sets at lower levels). For he explicitly denies that any sets in the multiverse have this property (since he takes each universe V to exist alongside a larger universe, which adds a subset to some set already contained in V). [15].

[14]He writes,"The toy model perspective can ultimately be unsatisfying, however, since it is of course in each case not the toy model in which we are interested, but rather the fully grown-up universe."[9]

[15]Relatedly, we might worry about whether Hamkins can develop an attractively unified account of what's required for interpretations of theorems in various set-theoretic axiom systems to count as having 'genuine set theoretic content'. An anonymous referee suggested Hamkins might do this by appealing to continuity with the history of prior set theory, maintaining that "what

For example, yes Hamkins can say forcing arguments tell us about ensembles of abstract objects (sets) which satisfy various axioms. But so do contemporary single-universe Platonists who take forcing arguments to tell us about how countable models of ZF can be extended (and whose attitude Hamkins compares to second stage pre-pluralism about geometry) [16]. One might object that the models conventional universe theorists acknowledge are not fully grown, because they are seen to be countable (and thus lacking some sets) from the perspective of some larger set universe.

However, *Hamkins also maintains* that every universe is countable from the perspective of some larger universe. Indeed, work by Hamkins and Gitman[8] shows that universe theorists should say there's a natural toy model (the collection of all countably computably saturated models of ZFC) which satisfies all the Hamkins' multiverse axioms in [10]. So, one might say, the mainstream pre-pluralist single universe set theorist *already* accepts every positive claim about iterative hierarchies of sets Hamkins' multiverse theorist wants to make, and only differs from the latter in claiming there's a largest overall universe which contains all these hierarchies. In this way, multiverse theory seems to be pluralism via leveling down (i.e., it secures geometrical pluralism merely by debunking traditional claims to unique favored status, without also attributing some substantive positive status to alternative understandings).

---

makes an axiom system of set-theoretic interest as opposed to mere formalist interest is that it is about the principles and phenomena which form the subject matter of set theory. So this would include the continuum hypothesis, forcing axioms, large cardinals, etc. This answer isn't provided by multiversism per se, but Hamkins's project is about explaining set-theoretic practice, so the multiversist can freely point to ordinary work in set theory without undermining themself". However, adopting this strategy would leave questions about what is required for something to count as e.g., giving a variant perspective on the continuum hypothesis (and therefore having genuine set theoretic content), if not all things which syntactically look to do so count? And one might also feel that it leaves us unable to vindicate intuitions that set theory has a single unified subject matter. It also would not seem to vindicate Hamkins claims that multiverse theory is distinctively able to see variant axioms as having full grown models.

[16] The completeness theorem ensures that all consistent first order theories have models. And according to single universe Platonism, these models are themselves sets and have abstract objects (sets) as their elements, so this means all syntactically consistent first order axioms can be interpreted as truly describing some abstract mathematical objects. Specifically single universe theorists take forcing arguments to tell us about the existence and extendability of countable models for the ZF axioms within the intended hierarchy of sets.

To put this point more generally[17], Hamkins maintains that every universe in the multiverse is countable (and, indeed, ill-founded) from the perspective of some other universe in the multiverse. This seems to suggest that every universe is a toy model, as seen by some larger universe. So how can Hamkins distinguish genuine interpretations from toy models?

Arguably, the real difference between Hamkins' pluralist multiverse theory and conventional non-pluralist understandings of set theory (which he compares to the stage two pre-pluralist attitudes to geometry) is only this. The mainstream single universe set theorist maintains that the true hierarchy of sets has certain kinds of positive status (e.g., being the fattest universe, containing sets corresponding to all possible ways of choosing subsets from sets it contains). In contrast, Hamkins' multiverse proposal denies that any model of set theory has these features[18].

If so, then multiverse Platonism actually turns out to be a form of leveling *down* pluralism (just as much as formalism is). For, switching to multiverse theory doesn't actually involve any positive element (of coming to see variant axiom systems as telling you about fully grown set theoretic universes rather than merely about countable models). Rather, Hamkins' multiverse proposal secures pluralism merely by *denying* that any interpretation of 'set' and 'element' had the features traditionally supposed to distinguish the intended model of set theory. So it would seem to be pluralism by debunking. Therefore we should question whether multiverse theory can get any special motivation (beyond that given to all forms of set theoretic pluralism) from calls to mirror (leveling up) pluralism about geometry.

5. What might Set Theoretic Pluralism via Leveling Up Look Like?

5.1. Basic Proposal for Leveling Up Pluralism about Set Theory.

If Hamkins' multiverse isn't (in any obvious way) an example of leveling up pluralism about set

[17] Thanks to a referee for suggesting this way of putting things, and reference to the theorem above.

[18] Note that the multiverse as a whole is not supposed to be an intended interpretation for set theory, and seemingly does not let you provide one.

theory, is such pluralism possible? What might real leveling up pluralism about set theory look like?

In this (final) section I'll argue that closely mirroring the example of leveling up geometrical pluralism in §3 yields an interesting variant on Hamkins' multiverse proposal, which *would* count as leveling up pluralism about set theory and could let us attractively:

- explicate what's required to see a theorem as having 'genuine set theoretic character'
- justify (a version of) Hamkins' claims to see variant axioms as telling us about "full-grown models" of set theory, in some distinctive sense.

Recall that in the case of geometry, we appealed to traditionally expected physical applications of geometry to help flesh out leveling up pluralism as follows. Traditionally, geometrical facts were expected to track facts about the structure of space and the possible spatial relationships between physical objects, by way of certain a priori expected bridge laws connecting geometry to physical facts[19]. We then identified coming to see theorems as 'having genuine geometrical content' with coming to see them as telling us about a conceivable way the structure of space could be, via these traditionally expected applications.

Can we mirror this in the case of set theory? Some nice parallels between a priori expected physical applications of set theory and geometry suggest a natural way to do this. However, while in the geometrical case we appealed to an intuition that there are different (metaphysically possible) ways physical space could be, in the set-theoretic case we'll consider different (metaphysically impossible but 'weakly' a priori epistemic[20]) options for facts about 'all possible ways of choosing'.

[19]For example someone holding this traditional view might say that geometrical claims commit us to facts about physical space because they were expected to come out true on all sufficiently intended/natural physical interpretations of 'point' and 'line'. And these latter facts about spatial points and lines, in turn, imply constraints on spatial relationships between objects by way of other a priori principles.

[20]Here I specify 'weakly' a priori epistemic live options, because a perfectly rational being would presumably have direct access to all facts about logical possibility/all possible ways of choosing.

To explain this in more detail, recall that on the traditional single universe view of set theory, there's a unique intended hierarchy of sets which satisfies the following iterative hierarchy conception. The intended hierarchy of sets has a well-ordered spine of ordinals. And at each ordinal layer there are sets available corresponding to 'all possible ways of choosing' some objects (either sets or ur-elements) available at lower layers [21]).

Accordingly, there's an a priori expected close relationship between set theory and lawlike (counterfactual-supporting) constraints on all possible ways of choosing how any properties and relations can apply to any objects. And this parallels the traditionally expected close relationship between geometry and lawlike, counterfactual-supporting constraints on spatial possibility (how physical objects can stand in spatial relations to one another) as follows.

- Naive geometry attempts to study what we might call spatial possibility – how it's (in some sense) possible for physical objects, points and lines to be spatially related to one another, via the idea that
  - The true principles of geometry will express truths on the intended physical interpretations of 'point' and 'line' etc.
  - Such facts about physical points and lines imply counterfactual-supporting constraints on the structure of space and hence how physical objects can relate to each other.
- Traditional (iterative hierarchy) set theory attempts to study how it's (in some sense) possible to choose how arbitrary properties apply to some objects (be they sets or ur-elements) via the idea that

In contrast, there are plausibly a range of options that cannot be ruled by the kind of logical and combinatorial principles and inference methods which we (very much non-ideal) beings are lucky enough to find a priori compelling (c.f. [2, 15].

[21] c.f.[4, 11]

– The true hierarchy of sets with ur-elements contains sets corresponding to 'all possible ways of choosing' (some physical objects, or some elements from a single set which this hierarchy contains).

– And such facts about all possible ways of choosing reflect constraints on how physical properties can apply to physical objects.

Considering this parallel suggests a direct way of mirroring the leveling up pluralism about geometry advocated in §3, in the case of set theory. We can say that

- **Actually Living in** a world corresponding to some set theoretic axioms[22] means living in a world where the intended hierarchy of sets (which contains sets corresponding to 'all possible ways of choosing' subsets from sets it contains etc.) obeys those axioms.
- Set theory (insofar as it's a branch of pure mathematics) equally studies all (weakly) a priori epistemic possibilities for how the facts about all possible ways of choosing (and thus all intended width hierarchies of sets) could be.
- Seeing work in variant axiom systems (whether studied directly or via forcing arguments) as genuinely set-theoretic involves seeing this work as revealing facts about some weakly a priori epistemic live option for how facts about all possible ways of choosing (and hence the intended hierarchy of sets) could be.

An advocate of this kind of leveling up set-theoretic pluralism could endorse a version of Hamkins' claim to see forcing arguments as telling us about full grown models (in a way that the mainstream single universe set theorists cannot). For this leveling up pluralist can say that fully grown models of set theory are ones which witness facts about all possible ways of choosing in the traditionally expected way (given by the iterative hierarchy conception of sets). And we can say that forcing arguments tell us about what would be true in the metaphysically impossible

[22]Here I mean to invoke the strong sense of 'living in' which Hamkins contrasts with merely imagining living in such a world.

scenarios where (facts about all possible ways of choosing/logical possibility were different so that) *these intended/fully grown* models of set theory had a different structure. Specifically, we can see the study of variant axiom systems via forcing arguments as telling us about metaphysically impossible scenarios where the laws of logical possibility are different (from those relevant to the actual world, or the model of set theory we work in when making the forcing argument) in the following way.

> An intrinsic duplicate of the (actual world) intended hierarchy of sets V (i.e., the thing we directly refer to and reason about when making this forcing argument) exists within a larger structure V[G]. V[G] (rather than V) is the intended model of set theory in this counterpossible scenario (so it contains sets witnessing 'all possible ways of choosing' *in the expanded sense relevant to this impossible world*).[23].

Even more specifically, we might see forcing arguments as telling us about counterpossible counterfactuals: what would be true if (per impossibile) the modal facts about logical possibility were different so that

- there are were extra logically possible ways of choosing some of the natural numbers (and sets witnessing this) than there actually are such that
- an intrinsic duplicate of the actual-world intended hierarchy of sets (which has the intended relationship to logical possibility facts in the actual world) can exist inside a larger intended hierarchy of sets (which reflects the more generous facts about 'all possible ways of choosing' in this scenario)

[23]For these purposes I'm thinking about laws in a non-Humean fashion, so that two possible (or impossible) scenarios can differ in what they make physically possible (and perhaps what they make logically possible) without differing in the actual non-model Humean mosaic of events. I'm also thinking about logical possibility/all possible ways of choosing as a fundamentally modal notion, which may or may not be elegantly and exhaustively described by any conveniently stateable list of 'laws of combinatorics'.

5.2. **Prior Work on Counterpossible Counterfactuals.** Appeal to conditionals with metaphysically impossible antecedents (like those above) might seem odd or problematic. But various tools for understanding such strange claims ('if the laws of logical possibility were different so as to allow extra subsets of the natural numbers as per ...forcing extension then...') have been suggested by prior work on impossible worlds and counter-possible counterfactuals[12].

Traditionally much work on counterfactuals has focused on claims with a false but metaphysically possible antecedent like 'if Nixon had pushed the button, then there would have been nuclear war'. Lewis and Stalnacker famously analyzed such claims by appealing to metaphysically possible worlds, with a counterfactual like this being true (at the actual world) if at all sufficiently close possible worlds to the actual world where the antecedent is true (Nixon pushed the button), the consequent is also true (nuclear war occurs). And different theories have been explored about how to understand the relevant closeness relation, with agreement that things like similarity to the actual world in important respects and mostly preserving general physical laws (except perhaps for minor miracles as needed to make the antecedent of the counterfactual true) tend to make for closeness.

One reason talk of metaphysically impossible worlds and counterpossibles has been unpopular is that according to Lewis' influential but infamous modal realism objects in possible worlds are just as real as objects in the actual world and have all properties attributed to them. So accepting non-trivial counter-possible counterfactuals would seem to require accepting that objects with metaphysically impossible combinations of properties (e.g., round square tables and marbles that are both red and not red) exist.

However, as Nolan [12] points out, most popular ways of making sense of possible worlds and counterfactual talk differ from Lewis' infamous modal realism in this regard. They don't require us to accept objects with metaphysically possible but not actual combinations of properties like, "talking donkeys, phlogiston, crystal spheres spinning around the center of their universe". Accordingly, they can often be naturally extended to allow for non-trivial counter-possible counterfactuals without commitment to any objects having metaphysically impossible combinations of properties.

For example, those who take possible worlds to be collections of propositions, or sets of sentence-like representations, can similarly identify impossible worlds with less restricted collections of propositions (which don't have to be logically coherent or metaphysically compatible). Those who take possible worlds to be sui generis abstract objects can say the same about impossible worlds. And those who do not admit the literal existence of possible worlds, but engage in talk of them all the same (fictionalists, instrumentalists , Meinongians) could do the same with impossible worlds.

Admittedly, questions arise about how to develop each approach. For example, it's not clear that taking possible worlds to be collections of sentences allows us to distinguish enough different scenarios, or make sense of talk about intrinsic duplicates of structures in one world existing within another. However, talk about intrinsic duplicates of objects at one world within another, is a staple of Lewissian metaphysics. And counterfactuals like 'if Gengis Khan had had two more children than he actually did. . . ' 'if Napoleon had been taller than he actually was. . . .' are widely accepted as potentially meaningful and nontrivial. So, one might argue, this is a problem for everyone (or all friends of counterfactuals).

Similarly, taking there to be sui generis abstract objects corresponding to different metaphysically impossible scenarios involving different iterative hierarchy structures raises cardinality paradox/Burali forti worries (however many objects there are, it would seem there should be more distinct possibilities). However one can raise the same cardinality worries about metaphysically possible worlds (if one does not think there's any metaphysically necessary limit on the number of objects that exist).

But if relevant metaphysical and technical obstacles to this project can be surmounted[24], we might come to see forcing arguments etc. as providing relatively concrete, precise, motivated and rigorous way of reasoning about metaphysically impossible scenarios and counter-possible counterfactuals.

5.3. **Features of this view.** Thinking about forcing arguments (and other ways of switching between universes) in this way, plausibly counts as set-theoretic pluralism by leveling up. For it lets us attribute work in variant axiom systems as having a substantive positive status (telling us about a cogent live option for what entwined facts about logical possibility and set theory would be like) not attributed to them by common contemporary pre-pluralist set theory, which regards such theorems as merely telling us about unintended models within the intended hierarchy of sets. We can perhaps accommodate Hamkins' remarks about phenomenology, his suggestion that forcing arguments seem to consider the whole hierarchy of sets you are explicitly working in (not just some countable model of ZFC inside it) being extended.

What about the other kinds of closure principles (aside from taking forcing extensions) that Hamkins suggests? To most directly parallel Hamkins' multiverse theory[25] (while providing genuine leveling up pluralism in the way I'm suggesting), the leveling up set theoretic pluralist could simply add that all of Hamkins' closure principles for the multiverse preserve cogency/being a weakly a priori live option in

[24]Admittedly, if we see forcing arguments as telling us about counterpossible counterfactuals in the way that I've suggested, we will have to say something a bit more circuitous about how making forcing arguments can establish consistency results (like the famous result that ZFC $+\neg CH$ is consistent). For there will be (remote) impossible worlds corresponding to syntactically inconsistent theories. So merely establishing that$\neg CH$ something would be true in the closest metaphysically impossible scenario where the hierarchy of sets in the actual world is extended in certain ways does not automatically give us reason to think that a theory which combines CH and ZFC is consistent. But one might fall back on the traditional realist way of getting from forcing to syntactic consistency to ensure this (i.e., one could say that forcing arguments *also* tell one about how countable models could be extended).
One might also worry about whether the (unquestioned, unreconstructed) talk of intrinsic duplicates I've indulged in creates pressure to accept claims about isomorphic mappability which a multiverse theorist should reject. I take no position on whether this problem is solvable, here.
[25]My leveling up pluralist identifies seeing a proof as having genuine set theoretic content, with seeing it as illuminating a (weakly) a priori live hypothesis about all possible ways of choosing (and hence the intended hierarchy of sets). They could, in principle, take a range of different positions about which axioms systems theories can reasonably be so interpreted.

the relevant sense (e.g., for every scenario that can be so considered, there's another one from whose perspective the first scenario is countable, not well founded etc.). [26]

Let me end by noting three things. First, although it is (ultimately) not my preferred view, I think the above proposal closely resembles Hamkins' multiverse in some interesting ways – and can be seen as the result of applying a small twist to Hamkins' original multiverse proposal which brings out the latent radicalness of the latter view.

For as we have noted, it is traditional to expect a certain close relationship between set theory and facts about all possible ways of choosing/logical possibility (and thereby constraints on how physical properties like color can apply to concrete objects like cats and spaceships). The set theoretic leveling up pluralism I've sketched results from simply keeping this connection between math and logic intact while embracing Hamkins' pluralism about set theory and claims to see forcing arguments as telling us about 'full grown' set theoretic universes. As a result, it sees pluralist set theory as not just exploring different set theoretic universes, but also scenarios where facts about 'all possible ways of choosing'/logical possibility (and thence plausibly also physical possibility) are different. The result is unsurprisingly radical and perhaps a bit psychedelic.

Just as we can make sense of (so to speak) Lovecraftian science fiction novels where protagonists' physical space has a radically different geometry, this leveling up pluralism about set theory suggests a way to make sense of Borghesian science fiction novels [13, 5] which depict life in (impossible) worlds with radically different set theory, logical possibility/'all possible ways of choosing' and perhaps also physical possibilities for concrete objects. And it (i.e., the leveling up set theoretic pluralism

[26]Possibilities which differ on the *width* of the intended hierarchy of sets, like CH, will correspond to different views about modal facts about all possible ways of choosing. For facts about all possible way of choosing fix facts about the intended interpretation of second order quantification and thereby - via Zermelo's quasicategoricity theorem - facts about the structure of the intended hierarchy of sets up to height.

sketched in this section) regards the study of such strange scenarios as a legitimate part of set theory.

Second, the version of leveling up pluralism sketched in this section may fortuitously help answer a puzzle about reference and multiverse theory raised by Barton in [1]. Hamkins talks about starting by working in one set theoretic universe and then repeatedly switching references by taking forcing extensions, considering ground models etc. But Barton asks how we could ever succeed in referring to a unique set universe if multiverse theory is true. After all, any axioms we assert about the sets will be satisfied by many universes. I don't think it would be a disaster for Hamkins to say that talk of 'the universe you are currently working in' is not meant seriously and there isn't actually a unique universe we are currently working in. However, a fan of the leveling up pluralism sketched in this section could instead answer Barton's challenge (at least as regards the width of the hierarchy of sets) in a more positive way. They could say that we, by default, initially refer to a set universe whose width reflects facts about logical possibility/all possible ways of choosing in the actual world. Forcing arguments could then be seen as revealing counter-possible counterfactuals, about what would be true if there were more possible ways of choosing so (intrinsic duplicates of) this actual-world-intended universe could exist within a larger one.

Third, considering this kind of leveling up pluralism calls into question how much analogies with geometry (alone) can do to motivate Hamkins' multiverse proposal – specifically his claim that there's no right answer to questions like CH, aside from mathematicians' choice of which axioms to work in. For my leveling up pluralist about set theory will presumably say there *is* a mathematical axiom choice independent right answer to CH (and all other questions determined entirely by the width of the hierarchy of sets) which reflects facts about all possible ways of choosing in the actual world – just as there's a physically right answer to the parallel postulate which reflects facts about spatial possibility in the actual world (what axioms come out true under all sufficiently intended/natural physical interpretations). And the leveling up pluralist will agree with Hamkins that mathematicians can legitimately study variant axiom systems and the results they prove will have genuine set theoretic content. But they will maintain that, in at least in one important sense there's a right answer to CH (reflecting actual-world facts about 'all possible ways of choosing'/logical possibility), whether or not we can ever discover it.

6. Conclusion

In this paper, I've argued that Hamkins'[10] special development of familiar pluralist analogies between set theory and geometry suggests a distinction between leveling up and leveling down versions of mathematical pluralism, which is interesting in its own right. On the unusual leveling up approach Hamkins seems to favor, we become pluralists partly by developing a substantive new way of interpreting work in variant axiom systems — not merely by debunking assumptions that favored certain choices of axioms.

I've suggested a way of fleshing out leveling up pluralism about geometry (inspired by Hamkins' brief remarks). I've then argued that it's not clear that Hamkins' Platonist multiverse actually delivers leveling up pluralism about set theory – calling into question his use of analogies between set theory and geometry to motivate multiverse theory (rather than mere pluralism). I've ended by giving an example of what genuine leveling up pluralism about set theory might look like (paralleling the above leveling up pluralism about geometry).

References

[1] Neil Barton. Multiversism and Concepts of Set: How Much Relativism Is Acceptable? In Francesca Boccuni and Andrea Sereni, editors, *Objectivity, Realism, and Proof. Filmat Studies in the Philosophy of Mathematics*, pages 189–209. Springer Verlag, 2016.

[2] Sharon Berry. Default Reasonableness and the Mathoids. *Synthese*, 190(17):3695–3713, 2013.

[3] Sharon Berry. *A Logical Foundation for Potentialist Set Theory*. Cambridge University Press, Cambridge, 2022.

[4] George Boolos. The Iterative Conception of Set. *Journal of Philosophy*, 68(8):215–231, 1971.

[5] Jorge Luis Borges and Andrew Hurley. *Collected fictions*. Penguin Classics Deluxe Edition. Penguin Books, New York, NY, 1998.

[6] Justin Clarke-Doane. *Morality and Mathematics*. Oxford University Press, 2020.

[7] Albert Einstein. Geometry and Experience. In *The Collected Papers of Albert Einstein, Volume 7: The Berlin Years: Writings, 1918-1921*, volume 7, pages 232–246. Princeton University Press, July 2002.

[8] Victoria Gitman and Joel David Hamkins. A Natural Model of the Multiverse Axioms. *Notre Dame Journal of Formal Logic*, 51(4):475–484, October 2010.

[9] Joel Hamkins. The multiverse perspective in set theory, August 2013.

[10] Joel David Hamkins. The Set-Theoretic Multiverse. *The Review of Symbolic Logic*, 5(3):416–449, September 2012.

[11] Vann McGee. How We Learn Mathematical Language. *Philosophical Review*, 106(1):35–68, 1997.

[12] Daniel Nolan. Impossible Worlds: A Modest Approach. *Notre Dame Journal of Formal Logic*, 38(4):535–572, 1997.

[13] Graham Priest, Koji Tanaka, and Zach Weber. Paraconsistent Logic. In Edward N. Zalta, editor, *The Stanford Encyclopedia of Philosophy*. Metaphysics Research Lab, Stanford University, winter 2016 edition.

[14] Wikipedia Editors. Geometry - Wikipedia, July 2023.

[15] Timothy Williamson. Absolute Provability and Safe Knowledge of Axioms. In Leon Horsten and Philip Welch, editors, *Gödel's Disjunction: The scope and limits of mathematical knowledge*. Oxford University Press, August 2016.